

A quantum Mermin–Wagner theorem for quantum rotators on 2D graphs

Mark Kelbert

Department of Mathematics, Swansea University, UK
IME, University of São Paulo, Brazil
m.kelbert@swansea.ac.uk

Yurii Suhov

StatsLab, DPMMS, University of Cambridge, UK
IME, University of São Paulo, Brazil
ITP, RAS, Moscow, Russia
yms@statslab.cam.ac.uk

Abstract. This is the first of a series of papers considering symmetry properties of quantum systems over 2D graphs or manifolds, with continuous spins, in the spirit of the Mermin–Wagner theorem [13]. In the model considered here (quantum rotators) the phase space of a single spin is a d –dimensional torus M , and spins (or particles) are attached to sites of a graph (Γ, \mathcal{E}) satisfying a special bi-dimensionality property. The kinetic energy part of the Hamiltonian is minus a half of the Laplace operator $-\Delta/2$ on M . We assume that the interaction potential is C^2 -smooth and invariant under the action of a connected Lie group \mathbf{G} (i.e., a Euclidean space $\mathbb{R}^{d'}$ or a torus M' of dimension $d' \leq d$) on M preserving the flat Riemannian metric. A part of our approach is to give a definition (and a construction) of a class of infinite-volume Gibbs states for the systems under consideration (the class \mathfrak{G}). This class contains the so-called limit Gibbs states, with or without boundary conditions. We use ideas and techniques originated from papers [3], [14], [4], [22] and [7], in combination with the Feynman–Kac representation, to prove that any state lying in the class \mathfrak{G} (defined in the text) is \mathbf{G} -invariant. An example is given where the interaction potential is singular and there exists a Gibbs state which is not \mathbf{G} -invariant.

In the next paper under the same title we establish a similar result for a bosonic model where particles can jump from a vertex $i \in \Gamma$ to one of its neighbors (a generalized Hubbard model).

Key words and phrases: quantum bosonic system with continuous

spins, symmetry group, the Feynman–Kac representation, bi-dimensional graphs, FK-DLR states, reduced density matrix (RDM), RDM functional, invariance

AMS Subject Classification: 82B10, 60B15, 82B26

1. Introduction. Existence and invariance of a limiting Gibbs state

This work had been motivated, on the one hand, by a spectacular success on Mermin–Wagner type theorems achieved in the past for a broad class of two-dimensional classical and quantum systems (see the bibliography quoted below) and, on the other hand, by a recognised progress in experimental quantum physics creating and working with thin materials like graphene. The main dissatisfaction with published rigorous results in this area stems for us from the fact that a natural class of quantum models remained uncovered. These are systems where the Hamiltonian contains a kinetic energy part given by a Laplacian. A serious problem here is that the finite-volume Hamiltonians are unbounded operators. As a result, the construction of the infinite-volume dynamical group encounters difficulties (it works fine for simplified quantum spins models (like Heisenberg’s) where the phase space of a spin is finite-dimensional). Consequently, the KMS-definition of an infinite-volume Gibbs state lacks substance for this class of models, apart from the non-interacting case. (At least this is the situation as we know it at the time of writing these lines.) A consistent definition of an infinite-volume Gibbs state is a cornerstone for the concept of a phase transition (as a non-uniqueness phenomenon); it is precisely this concept that makes the Mermin–Wagner theorem important (and elegant).

1.1. Bi-dimensional graphs. In the present paper we focus on Mermin–Wagner type result for a quantum bosonic system with continuous spins, over a denumerable graph (Γ, \mathcal{E}) (with a vertex set Γ and an edge set $\mathcal{E} \subset \Gamma \times \Gamma$). The graph will be assumed to satisfy a specific bi-dimensional property generalising properties of ‘regular’ lattices such as a square lattice \mathbb{Z}^2

or a triangular lattice \mathbb{Z}_{Δ}^2 . Cf. Eqns (1.1.1), (1.1.2) below. (Graphene is clearly a regular 2D lattice; however, the whole theoretical methodology could be examined in the context of a more general graph with a distinct bi-dimensionality property.) More precisely, we assume that (Γ, \mathcal{E}) has the property that whenever edge $(j', j'') \in \mathcal{E}$, the reversed edge $(j'', j') \in \Upsilon$ as well. Furthermore, (Γ, \mathcal{E}) is without multiple edges and has a bounded degree. The latter means that the number of edges (j, j') with a fixed initial or terminal vertex is uniformly bounded:

$$\sup \left[\max \left(\# \{j' \in \Gamma : (j, j') \in \Upsilon\}, \right. \right. \\ \left. \left. \# \{j' \in \Gamma : (j', j) \in \mathcal{E}\} \right) : j \in \Gamma \right] < \infty. \quad (1.1.1)$$

The bi-dimensionality property is expressed in the bound

$$0 < \sup \left[\frac{1}{n} \# \Sigma(j, n) : j \in \Gamma, n = 1, 2, \dots \right] < \infty \quad (1.1.2)$$

where $\Sigma(j, n)$ denotes the set of vertices in Γ at graph distance n from site $j \in \Gamma$ (a sphere of radius n about j):

$$\Sigma(j, n) = \{j' \in \Gamma : d(j, j') = n\}. \quad (1.1.3)$$

(The graph distance $d(j, j') = d_{\Gamma, \mathcal{E}}(j, j')$ between sites $j, j' \in \Gamma$ is defined as the minimal length of a path on (Γ, \mathcal{E}) joining j and j' .) This implies that the cardinality of the ball

$$\Lambda(j, n) = \{j' \in \Gamma : d(j, j') \leq n\}. \quad (1.1.4)$$

grows at most quadratically in n .

1.2. The phase space and the group action. We consider the following model. With each site (vertex) $j \in \Gamma$ there is associated a Hilbert space \mathcal{H} realized as $L_2(M, v)$ where M is a compact Riemannian manifold; v stands for the induced Riemannian volume. In this paper we assume that M is a d -dimensional torus $\mathbb{R}^d / \mathbb{Z}^d$. However, parts of the argument which can be easily done for a general manifold are conducted without referring to the specific case of the torus. (The full generalization of the main results for a general compact Riemannian manifold will be discussed elsewhere.) Physically, \mathcal{H} is the phase space of a quantum spin ‘attached’ to a single site

of the graph and M is its classical prototype. We assume that a connected Lie group \mathbf{G} is given, acting on M and preserving the flat metric on M . Transitivity of the action is not needed, hence \mathbf{G} is itself a torus or a Euclidean space of dimension $d' \leq d$. The action is generally referred to as

$$(\mathbf{g}, x) \in \mathbf{G} \times M \mapsto \mathbf{g}x \in M. \quad (1.2.1)$$

An alternative is the additive form of writing: we represent an element $\mathbf{g} \in \mathbf{G}$ with a d' -dimensional vector

$$\underline{\theta} = \theta A$$

where $\theta \in \mathbb{R}^{d'}/\mathbb{Z}^{d'}$ is a vector of dimension d' and A is a $d' \times d$ matrix with rational coefficient of the column rank d' . The action is then written as

$$(\underline{\theta}, x) \mapsto x + \underline{\theta} \bmod 1. \quad (1.2.2)$$

We will use both forms: the multiplicative form (1.2.1) makes formulas shorter whereas the additive one is more convenient in technical calculations.

A physical example of a system of the above type is a ‘frustrated’ 2D crystal lattice. Here some ‘heavy’ atoms or ions are placed at the vertices of a graph, and each atom possesses a light bosonic particle moving according to standard rules of Quantum Mechanics. A more complicated model arises when the number of particles is not fixed, and they can ‘jump’ from one vertex to another; see [8].

Another example emerges from quantum gravity: cf. [9], [10]. Here, a graph is random and emerges from (random) triangulations of a $1 + 1$ -dimensional space-time complex. (The paper [9] deals with classical spins on random triangulations; a quantum version of the model is treated in [10].) Classical models on general graphs with a variable structure have been treated in a recent paper [11].

If Λ is a finite subset in Γ then the phase space of the quantum system over Λ is $\mathcal{H}_\Lambda := \mathcal{H}^{\otimes \Lambda}$, the Hilbert space $L_2(M^\Lambda, v^\Lambda)$. Here and below the superscripts $\otimes \Lambda$ and Λ mean, respectively, the tensor product of copies of $\mathcal{H} = L_2(M, v)$ and the Cartesian products of copies of M and v , labelled by sites $j \in \Lambda$. Formally, elements of $\mathcal{H}^{\otimes \Lambda}$ are (complex) functions

$$\mathbf{x}_\Lambda = (x(j), j \in \Lambda) \in M^\Lambda \mapsto \phi(\mathbf{x}_\Lambda) \in \mathbb{C}$$

considered modulo a set of v^Λ -measure 0, with the standard norm and the

scalar product

$$\|\phi\| = \left(\int_{M^\Lambda} |\phi(\mathbf{x}_\Lambda)|^2 \prod_{j \in \Lambda} v(dx(j)) \right)^{1/2},$$

and

$$\langle \phi_1, \phi_2 \rangle = \int_{M^\Lambda} \phi_1(\mathbf{x}_\Lambda) \overline{\phi_2(\mathbf{x}_\Lambda)} \prod_{j \in \Lambda} v(dx(j)).$$

The argument $\mathbf{x}_\Lambda \in M^\Lambda$ represents a classical configuration of particles in Λ . Physically, this setting leads to a bosonic nature of the models under consideration.

The action of \mathbb{G} determines unitary operators $U_\Lambda(\mathbf{g})$, $\mathbf{g} \in \mathbb{G}$, in \mathcal{H}_Λ :

$$U_\Lambda(\mathbf{g})\phi(\mathbf{x}_\Lambda) = \phi(\mathbf{g}^{-1}\mathbf{x}_\Lambda) \text{ where } \mathbf{g}^{-1}\mathbf{x}_\Lambda = \{\mathbf{g}^{-1}\mathbf{x}(j), j \in \Lambda\}. \quad (1.2.3)$$

1.3. The Hamiltonian of the model and assumptions about the potential. A standard form of the kinetic energy operator for an individual spin is $-\Delta/2$ where Δ stands for the Laplacian operator in \mathcal{H} . We also assume that a two-body interaction potential is given, which is described by a real-valued function

$$((x', j'), (x'', j'')) \mapsto J(\mathbf{d}(j', j''))V(x', x''). \quad (1.3.1)$$

In the main body of the paper we assume that the (real) function $(x', x'') \in M \times M \mapsto V(x', x'')$ is of class C^2 , although in one particular result, Theorem 1.4, we consider an ‘opposite’ situation of a singular potential. (In a forthcoming paper, we will address in detail the case of quantum models with non-smooth potentials.)

More precisely, in Theorems 1.1, 1.2, 3.1–3.2 and Corollary 3.3 below we assume that the function V and its first and second derivatives $\nabla_{\mathbf{x}}V$ and $\nabla_{\mathbf{x}_1}\nabla_{\mathbf{x}_2}V$ satisfy the uniform bounds: $x', x'' \in M$

$$|V(x', x'')|, |\nabla_{\mathbf{x}}V(x', x'')|, |\nabla_{\mathbf{x}_1}\nabla_{\mathbf{x}_2}V(x', x'')| \leq \overline{V}. \quad (1.3.2)$$

Here \mathbf{x} , \mathbf{x}_1 and \mathbf{x}_2 run through the arguments $x', x'' \in M$; $|\cdot|$ stands for the absolute value of a real scalar or the norm of a real vector, and $\overline{V} \in (0, +\infty)$ is a constant. Next, the function $J : r \in (0, \infty) \mapsto J(r) \geq 0$ is assumed

monotonically non-increasing with r and obeying the relation $\overline{J}(l) \rightarrow 0$ as $l \rightarrow \infty$ where

$$\overline{J}(l) = \sup \left[\sum_{j'' \in \Gamma} J(\mathbf{d}(j', j'')) \mathbf{1}(\mathbf{d}(j', j'') \geq l) : j' \in \Gamma \right] < \infty. \quad (1.3.3)$$

Additionally, let the interaction potential be such that

$$J^* = \sup \left[\sum_{j'' \in \Gamma} J(d(j', j'')) d(j', j'')^2 : j' \in \Gamma \right] < \infty. \quad (1.3.4)$$

Next, we assume that the function V is \mathbf{g} -invariant:

$$V(x, x') = V(\mathbf{g}x, \mathbf{g}x'), \quad \forall x, x' \in M, \mathbf{g} \in \mathbf{G}. \quad (1.3.5)$$

The Hamiltonian H_Λ of the system over a finite set $\Lambda \subset \Gamma$ acts on functions $\phi \in \mathcal{H}^{\otimes \Lambda}$: given $\mathbf{x}_\Lambda = (x(j), j \in \Lambda) \in M^\Lambda$,

$$(H_\Lambda \phi)(\mathbf{x}_\Lambda) = \frac{1}{2} \left[- \sum_{j \in \Lambda} \Delta_j + \sum_{j, j' \in \Lambda \times \Lambda} J(\mathbf{d}(j, j')) V(x_j, x_{j'}) \right] \phi(\mathbf{x}_\Lambda). \quad (1.3.6)$$

Here Δ_j stands for the Laplace operator in variable $x(j) \in M$. A more general concept is a Hamiltonian $H_{\Lambda|\overline{\mathbf{x}}_{\Gamma' \setminus \Lambda}}$ in the external field generated by a (finite or infinite) configuration $\overline{\mathbf{x}}_{\Gamma' \setminus \Lambda} = \{\overline{x}_{j'}, j' \in \Gamma' \setminus \Lambda\} \in M^{\Gamma' \setminus \Lambda}$ where $\Gamma' \subseteq \Gamma$ is a (finite or infinite) collection of vertices. Namely,

$$\begin{aligned} (H_{\Lambda|\overline{\mathbf{x}}_{\Gamma' \setminus \Lambda}} \phi)(\mathbf{x}_\Lambda) = & \left[- \frac{1}{2} \sum_{j \in \Lambda} \Delta_j + \frac{1}{2} \sum_{(j, j') \in \Lambda \times \Lambda} J(\mathbf{d}(j, j')) V(x_j, x_{j'}) \right. \\ & \left. + \sum_{(j, j') \in \Lambda \times (\Gamma' \setminus \Lambda)} J(\mathbf{d}(j, j')) V(x_j, \overline{x}_{j'}) \right] \phi(\mathbf{x}_\Lambda). \end{aligned} \quad (1.3.7)$$

Summarizing, the model considered in this paper can be called a system of quantum rotators on a bi-dimensional graph.

1.4. Properties of limiting Gibbs states. Throughout the paper, we use a number of well-known facts (properties (i)–(iv) and (a)–(c) below) related to operators H_Λ and $H_{\Lambda|\overline{\mathbf{x}}_{\Gamma' \setminus \Lambda}}$ which can be extracted, e.g., from Refs

[2], [6], [15], [20]. (i) Under the above assumptions, operators H_Λ and $H_{\Lambda|\overline{\mathfrak{X}}_{\Gamma'\setminus\Lambda}}$ are self-adjoint (on the natural domains) in $\mathcal{H}^{\otimes\Lambda}$, bounded from below and have a discrete spectrum. (ii) Moreover, $\forall \beta > 0$, H_Λ and $H_{\Lambda|\overline{\mathfrak{X}}_{\Gamma'\setminus\Lambda}}$ give rise to positive-definite trace-class operators $\exp [-\beta H_\Lambda]$ and $\exp [-\beta H_{\Lambda|\overline{\mathfrak{X}}_{\Gamma'\setminus\Lambda}}]$. (iii) In turn, this gives rise to Gibbs states $\varphi_\Lambda = \varphi_{\beta,\Lambda}$ and $\varphi_{\Lambda|\overline{\mathfrak{X}}_{\Gamma'\setminus\Lambda}} = \varphi_{\beta,\Lambda|\overline{\mathfrak{X}}_{\Gamma'\setminus\Lambda}}$, at temperature β^{-1} in volume Λ . These are linear positive normalized functionals on the C*-algebra \mathfrak{B}_Λ of bounded operators in space \mathcal{H}_Λ :

$$\varphi_\Lambda(A) = \text{tr}_{\mathcal{H}_\Lambda}(R_\Lambda A), \quad \varphi_{\Lambda|\overline{\mathfrak{X}}_{\Gamma'\setminus\Lambda}}(A) = \text{tr}_{\mathcal{H}_\Lambda}(R_{\Lambda|\overline{\mathfrak{X}}_{\Gamma'\setminus\Lambda}} A), \quad A \in \mathfrak{B}_\Lambda, \quad (1.4.1)$$

where

$$R_\Lambda = \frac{\exp [-\beta H_\Lambda]}{\Xi_{\beta,\Lambda}} \quad \text{with} \quad \Xi_{\beta,\Lambda} = \text{tr}_{\mathcal{H}_\Lambda}(\exp [-\beta H_\Lambda]) \quad (1.4.2)$$

and

$$R_{\Lambda|\overline{\mathfrak{X}}_{\Gamma'\setminus\Lambda}} = \frac{\exp [-\beta H_{\Lambda|\overline{\mathfrak{X}}_{\Gamma'\setminus\Lambda}}]}{\Xi_{\beta,\Lambda|\overline{\mathfrak{X}}_{\Gamma'\setminus\Lambda}}} \quad (1.4.3)$$

with $\Xi_{\beta,\Lambda|\overline{\mathfrak{X}}_{\Gamma'\setminus\Lambda}} = \text{tr}_{\mathcal{H}_\Lambda}(\exp [-\beta H_{\Lambda|\overline{\mathfrak{X}}_{\Gamma'\setminus\Lambda}}])$.

(iv) Let \mathfrak{B} stand for the C*-algebra of bounded operators in Hilbert space \mathcal{H} . For $\Lambda^0 \subset \Lambda$, the representations $\mathfrak{B}_\Lambda = \mathfrak{B}^{\otimes\Lambda}$ and $\mathfrak{B}_{\Lambda^0} = \mathfrak{B}^{\otimes\Lambda^0}$ identify \mathfrak{B}_{Λ^0} with the C*-sub-algebra in \mathfrak{B}_Λ formed by the operators of the form $A_0 \otimes I_{\Lambda\setminus\Lambda^0}$ where $I_{\Lambda\setminus\Lambda^0}$ is the unit operator in $\mathcal{H}_{\Lambda\setminus\Lambda^0}$. Accordingly, the restriction $\varphi_\Lambda^{\Lambda^0}$ of state φ_Λ to C*-algebra \mathfrak{B}_{Λ^0} is given by

$$\varphi_\Lambda^{\Lambda^0}(A_0) = \text{tr}_{\mathcal{H}_{\Lambda^0}}(R_\Lambda^{\Lambda^0} A_0), \quad A_0 \in \mathfrak{B}_{\Lambda^0}, \quad (1.4.4)$$

where

$$R_\Lambda^{\Lambda^0} = \text{tr}_{\mathcal{H}_{\Lambda\setminus\Lambda^0}} R_\Lambda. \quad (1.4.5)$$

Clearly, operators $R_\Lambda^{\Lambda^0}$ are positive-definite and have $\text{tr}_{\mathcal{H}_{\Lambda^0}} R_\Lambda^{\Lambda^0} = 1$. They also satisfy the compatibility property: $\forall \Lambda^0 \subset \Lambda^1 \subset \Lambda$,

$$R_\Lambda^{\Lambda^0} = \text{tr}_{\mathcal{H}_{\Lambda^1\setminus\Lambda^0}} R_\Lambda^{\Lambda^1}. \quad (1.4.6)$$

Furthermore, in a similar fashion one can define functionals $\varphi_{\Lambda|\overline{\mathfrak{X}}_{\Gamma'\setminus\Lambda}}^{\Lambda^0}$ and operators $R_{\Lambda|\overline{\mathfrak{X}}_{\Gamma'\setminus\Lambda}}^{\Lambda^0}$, with the same properties.

Below we denote by $\Lambda \nearrow \Gamma$ the net of finite subsets of Γ ordered by inclusion. A convenient example of an increasing sequence in this net, eventually covering the entire Γ , is formed by sets $\Lambda(j, n)$, $n = 1, 2, \dots$ (balls in the graph distance); see (1.1.4).

We prove in this paper the following results:

Theorem 1.1. *For all given $\beta \in (0, \infty)$ and a finite $\Lambda^0 \subset \Gamma$, operators $R_{\Lambda}^{\Lambda^0}$ form a compact sequence in the trace-norm topology in \mathcal{H}_{Λ^0} as $\Lambda \nearrow \Gamma$. Furthermore, given any family of (finite or infinite) sets $\Gamma' = \Gamma'(\Lambda) \subseteq \Gamma$ and particle configurations $\bar{\mathfrak{x}}_{\Gamma' \setminus \Lambda}$, operators $R_{\Lambda | \bar{\mathfrak{x}}_{\Gamma' \setminus \Lambda}}^{\Lambda^0}$ also form a compact sequence in the trace-norm topology.*

Moreover, any limiting point, R^{Λ^0} , for $\left\{ R_{\Lambda | \bar{\mathfrak{x}}_{\Gamma' \setminus \Lambda}}^{\Lambda^0} \right\}$ is a positive definite operator of trace one which possesses the following invariance property: $\forall \mathbf{g} \in \mathbb{G}$,

$$U_{\Lambda^0}(\mathbf{g})^{-1} R^{\Lambda^0} U_{\Lambda^0}(\mathbf{g}) = R^{\Lambda^0}. \quad \triangleleft \quad (1.4.7)$$

By invoking the diagonal process, we get a family $\{R^{\Lambda^0}\}$ of positive definite operators R^{Λ^0} in \mathcal{H}_{Λ^0} of trace one, where Λ^0 runs over finite subsets of Γ , with the following properties. (a) \exists an increasing sequence of finite sets $\Lambda_{n_k} \subset \Gamma$ such that $\bigcup_k \Lambda_{n_k} = \Gamma$ and a sequence of sets $\Gamma'_{n_k} \subseteq \Gamma$ and particle configurations $\bar{\mathfrak{x}}_{\Gamma'_{n_k} \setminus \Lambda_{n_k}}$ such that for all finite set Λ^0 the convergence in the trace-norm holds:

$$R^{\Lambda^0} = \lim_{k \rightarrow \infty} R_{\Lambda_{n_k} | \bar{\mathfrak{x}}_{\Gamma'_{n_k} \setminus \Lambda_{n_k}}}^{\Lambda^0}. \quad (1.4.8)$$

(b) \forall finite subsets Λ^0, Λ^1 of Γ , with $\Lambda^0 \subset \Lambda^1$,

$$R^{\Lambda^0} = \text{tr}_{\mathcal{H}_{\Lambda^1 \setminus \Lambda^0}} R^{\Lambda^1}. \quad (1.4.9)$$

(c) Such a family defines a state φ of (that is, a linear positive normalized functional on) the C^* -algebra quasilocal observables $\mathfrak{B}_{\Gamma} = (\mathfrak{B}_{\Gamma}^0)^-$. Here $*$ -algebra \mathfrak{B}_{Γ}^0 is the inductive limit $\lim_{n \rightarrow \infty} \mathfrak{B}_{\Lambda_n}$ and superscript $-$ in the notation $(\mathfrak{B}_{\Gamma}^0)^-$ stands for the norm completion. See [2].

The definition of the above state φ is that \forall finite $\Lambda^0 \subset \Gamma$,

$$\varphi(A) = \text{tr}_{\mathcal{H}_{\Lambda^0}} R^{\Lambda^0} A.$$

Reflecting its construction, we call φ a limiting Gibbs state; Theorem 1.1 asserts that the set \mathfrak{G}^0 of limiting Gibbs states is non-empty. A straightforward corollary is

Theorem 1.2. *Any limiting Gibbs state $\varphi \in \mathfrak{G}^0$ has the following invariance property: \forall finite $\Lambda^0 \subset \Gamma$ any $A \in \mathfrak{B}_{\Lambda^0}$ and $\mathbf{g} \in \mathbb{G}$,*

$$\varphi(A) = \varphi(U_{\Lambda^0}(\mathbf{g})^{-1} A U_{\Lambda^0}(\mathbf{g})). \quad \triangleleft \quad (1.4.10)$$

The proof of Theorem 1.1 is based on the following Lemma.

Lemma 1.1. *Let $\rho_n(x, y)$ be a sequence of kernels defining positive-definite operators R_n of trace class and with trace 1 in a Hilbert space $L_2(M, \nu)$ where $\nu(M) < \infty$. Suppose there exists the following limit, uniform in $x, y \in M$:*

$$\lim_{n \rightarrow \infty} \rho_n(x, y) = \rho(x, y), \quad (1.4.11)$$

which defines a positive-definite trace-class operator R of trace 1. Then

$$\lim_{n \rightarrow \infty} \|R_n - R\|_{\text{tr}} = 0 \quad (1.4.12)$$

where $\|A\|_{\text{tr}} = \text{tr}(AA^)^{1/2}$. \triangleleft*

Lemma 1.1 appeared for the first time in the short note [23]. For the reader's convenience we give a complete proof in Section 4.3.

Remark 1.1. As usually with Mermin–Wagner type assertions, Theorem 1.2 does not address the issue of phase transitions, viz., uniqueness of a limiting Gibbs state. A matter of principle here is to determine within what class of states $\mathfrak{G} \supseteq \mathfrak{G}^0$ the invariance property still holds true. Such a class is introduced in the next section; it is related to the Feynman–Kac representation of operator $\exp[-\beta H_\Lambda]$.

Throughout the paper we adopt the following notational agreement: symbol \triangleleft marks the end of a statement and symbol \square the end of a proof.

2. The Feynman–Kac formula and DLR equations

2.1. The Feynman–Kac (FK) representation for the partition function. In this section we follow the approach developed in [6]; see also [1]. Our first observation is that, under the above assumptions, operator $\exp [-\beta H_\Lambda]$ acts as an integral operator in variables $\mathbf{x}_\Lambda = (x(j), j \in \Lambda) \in M^\Lambda$ and $\mathbf{y}_\Lambda = (y(j), j \in \Lambda) \in M^\Lambda$:

$$\left(\exp [-\beta H_\Lambda] \phi \right) (\mathbf{x}_\Lambda) = \int_{M^\Lambda} \prod_{j \in \Lambda} v(dy(j)) K_{\beta, \Lambda}(\mathbf{x}_\Lambda, \mathbf{y}_\Lambda) \phi(\mathbf{y}_\Lambda). \quad (2.1.1)$$

The integral kernel $K_{\beta, \Lambda}(\mathbf{x}_\Lambda, \mathbf{y}_\Lambda)$ admits a Feynman–Kac (FK) integral representation

$$K_{\beta, \Lambda}(\mathbf{x}_\Lambda, \mathbf{y}_\Lambda) = \int_{\overline{W}_{\mathbf{x}_\Lambda, \mathbf{y}_\Lambda}^\beta} P_{\mathbf{x}_\Lambda, \mathbf{y}_\Lambda}^\beta(d\overline{\omega}_\Lambda) \exp [-h^\Lambda(\overline{\omega}_\Lambda)] \quad (2.1.2)$$

explained below.

In Eqn (2.1.2), $\overline{W}_{\mathbf{x}_\Lambda, \mathbf{y}_\Lambda}^\beta$ stands for the Cartesian product $\times_{j \in \Lambda} \overline{W}_{x(j), y(j)}^\beta$.

Next, the Cartesian factor $\overline{W}_{x(j), y(j)}^\beta$ represents the space of continuous paths $\overline{\omega}_j$ in M , of time-length β and with the end-points $x(j)$ and $y(j)$:

$$\begin{aligned} \overline{\omega}_j : \tau \in [0, \beta] &\mapsto \overline{\omega}_j(\tau) \in M : \overline{\omega}_j(\cdot) \text{ continuous,} \\ \overline{\omega}_j(0) &= x(j), \quad \overline{\omega}_j(\beta) = y(j), \quad j \in \Lambda. \end{aligned}$$

Correspondingly, $\overline{\omega}_\Lambda = (\overline{\omega}_j, j \in \Lambda) \in \overline{W}_{\mathbf{x}_\Lambda, \mathbf{y}_\Lambda}^\beta$ is a collection of continuous paths $\overline{\omega}_j \in \overline{W}_{x(j), y(j)}^\beta$, $j \in \Lambda$. We will say that $\overline{\omega}_\Lambda$ is a path configuration over Λ . Further, $P_{\mathbf{x}_\Lambda, \mathbf{y}_\Lambda}^\beta$ is the product-measure on $\overline{W}_{\mathbf{x}_\Lambda, \mathbf{y}_\Lambda}^\beta$:

$$P_{\mathbf{x}_\Lambda, \mathbf{y}_\Lambda}^\beta(d\overline{\omega}_\Lambda) = \times_{j \in \Lambda} P_{x(j), y(j)}^\beta(d\overline{\omega}_j) \quad (2.1.3)$$

where $P_{x(j), y(j)}^\beta(d\overline{\omega}_j)$ is the (non-normalised) Wiener measure on $\overline{W}_{x(j), y(j)}^\beta$ (the Brownian bridge in M , of time-length β , with endpoints $x(j)$ and $y(j)$). The measure $P_{x(j), y(j)}^\beta(d\overline{\omega}_j)$ is defined on a standard sigma-algebra of subsets

of $\overline{W}_{x(j),y(j)}^\beta$ generated by cylinders, and the measure $\mathbb{P}_{\mathbf{x}_\Lambda, \mathbf{y}_\Lambda}^\beta$ on the corresponding sigma-algebra of subsets in $\overline{W}_{\mathbf{x}_\Lambda, \mathbf{y}_\Lambda}^\beta$. In future we do not always explicitly refer to the sigma-algebras where measures under consideration are defined: their specification follows that of the underlying spaces.

Finally, for a path configuration $\overline{\omega}_\Lambda = \{\omega_j, j \in \Lambda\}$ over Λ ,

$$h^\Lambda(\overline{\omega}_\Lambda) = \sum_{(j,j') \in \Lambda \times \Lambda} h^{j,j'}(\overline{\omega}_j, \overline{\omega}_{j'}) \quad (2.1.4)$$

where $h^{j,j'}(\overline{\omega}_j, \overline{\omega}_{j'})$ represents an integral along trajectories $\overline{\omega}_j$ and $\overline{\omega}_{j'}$:

$$h^{j,j'}(\overline{\omega}_j, \overline{\omega}_{j'}) = J(\mathbf{d}(j, j')) \int_0^\beta d\tau V(\overline{\omega}_j(\tau), \overline{\omega}_{j'}(\tau)). \quad (2.1.5)$$

It is convenient to think that $h^{j,j'}(\overline{\omega}_j, \overline{\omega}_{j'})$ yields the ‘energy of interaction’ between trajectories $\overline{\omega}_j$ and $\overline{\omega}_{j'}$, and $h^\Lambda(\overline{\omega}_\Lambda)$ equals the ‘full potential energy’ of the path configuration $\overline{\omega}_\Lambda$.

Furthermore, the trace $\text{tr}_{\mathcal{H}_\Lambda} \exp [-\beta H_\Lambda]$ (the partition function in Λ) is finite and equals $\Xi_{\beta, \Lambda}$ where

$$\Xi_{\beta, \Lambda} = \int_{M^\Lambda} \prod_{j \in \Lambda} v(dx(j)) K_{\beta, \Lambda}(\mathbf{x}_\Lambda, \mathbf{x}_\Lambda) < +\infty. \quad (2.1.6)$$

Consequently, operator R_Λ from (1.4.2) (often called the density matrix (DM) in Λ) is given by its integral kernel $F_{\beta, \Lambda}(\mathbf{x}_\Lambda, \mathbf{y}_\Lambda)$ (the DM kernel, DMK for short):

$$F_{\beta, \Lambda}(\mathbf{x}_\Lambda, \mathbf{y}_\Lambda) = \frac{1}{\Xi_{\beta, \Lambda}} K_{\beta, \Lambda}(\mathbf{x}_\Lambda, \mathbf{y}_\Lambda). \quad (2.1.7)$$

2.2. The FK representation for the RDMK in a finite volume.

The operator $R_\Lambda^{\Lambda^0}$ from (1.4.8), (1.4.9) (referred to as a reduced DM, briefly, RDM) is determined by its own integral kernel $F_{\beta, \Lambda}^{\Lambda^0}(\mathbf{x}_{\Lambda^0}, \mathbf{y}_{\Lambda^0})$ (the RDM kernel, shortly, RDMK):

$$F_{\beta, \Lambda}^{\Lambda^0}(\mathbf{x}_{\Lambda^0}, \mathbf{y}_{\Lambda^0}) = \frac{\Xi_{\beta, \Lambda \setminus \Lambda^0}(\mathbf{x}_{\Lambda^0}, \mathbf{y}_{\Lambda^0})}{\Xi_{\beta, \Lambda}}, \quad \mathbf{x}_{\Lambda^0}, \mathbf{y}_{\Lambda^0} \in M^{\Lambda^0}. \quad (2.2.1)$$

Here the quantity $\Xi_{\beta, \Lambda \setminus \Lambda^0}(\mathbf{x}_{\Lambda^0}, \mathbf{y}_{\Lambda^0})$ in the numerator yields a ‘partial’ partition function corresponding to the partial trace $\text{tr}_{\mathcal{H}_{\Lambda \setminus \Lambda^0}}$ in (1.4.6):

$$\begin{aligned} \Xi_{\beta, \Lambda \setminus \Lambda^0}(\mathbf{x}_{\Lambda^0}, \mathbf{y}_{\Lambda^0}) &= \int_{M^{\Lambda \setminus \Lambda^0}} \prod_{j \in \Lambda \setminus \Lambda^0} v(dz_j) \\ &\times K_{\beta}(\mathbf{x}_{\Lambda^0} \vee \mathbf{z}_{\Lambda \setminus \Lambda^0}, \mathbf{y}_{\Lambda^0} \vee \mathbf{z}_{\Lambda \setminus \Lambda^0}), \end{aligned} \quad (2.2.2)$$

where symbol \vee means concatenation of configurations (this notation will be repeatedly used below).

It is convenient to use a brief notation $d\mathbf{x}_{\Lambda}$ for the product of the Riemannian volumes $\times v(dx(j))$. We will also omit, where possible, the argument/index β from the notation (viz., by writing Ξ_{Λ} instead of $\Xi_{\beta, \Lambda}$). The above representations (2.1.1)–(2.1.7) allow us to associate with Gibbs state φ_{Λ} a probability distribution μ_{Λ} on the set

$$W_{\Lambda} = \bigcup_{\mathbf{x}_{\Lambda} \in M^{\Lambda}} \overline{W}_{\mathbf{x}_{\Lambda}, \mathbf{x}_{\Lambda}} \quad \text{where} \quad \overline{W}_{\mathbf{x}_{\Lambda}, \mathbf{x}_{\Lambda}} = \bigcup_{j \in \Lambda} \overline{W}_{x(j), x(j)}. \quad (2.2.3)$$

the definition of this probability distribution is provided in forthcoming paragraphs.

Pictorially, W_{Λ} is the space of collections of closed trajectories (loops) in M issued from and returning to (coinciding) specified endpoints; each loop being assigned to a site $j \in \Lambda$. Any such loop collection can be written as a pair $(\mathbf{x}_{\Lambda}, \boldsymbol{\omega}_{\Lambda})$. Here $\boldsymbol{\omega}_{\Lambda} = \{\omega_j, j \in \Lambda\}$ is a collection of loops $\tau \in [0, \beta] \mapsto \omega_j(\tau)$, where $\omega_j(0) = \omega_j(\beta) = x(j)$; a pair $(x(j), \omega_j)$ is associated with site $j \in \Lambda$. We will say that $(\mathbf{x}_{\Lambda}, \boldsymbol{\omega}_{\Lambda})$ (and $\boldsymbol{\omega}_{\Lambda}$ when the reference to \mathbf{x}_{Λ} is clear from the context) is a loop configuration over Λ . Note the absence of the bar in this notation, stressing that $\boldsymbol{\omega}_{\Lambda} \in W_{\Lambda}$ is a loop configuration as opposite to a general path configuration $\overline{\boldsymbol{\omega}}_{\Lambda} \in \overline{W}_{\mathbf{x}_{\Lambda}, \mathbf{y}_{\Lambda}} \subset \overline{W}_{\Lambda}$ (again associated with sites $j \in \Lambda$ (see Eqn (2.2.4) below)). More precisely, when appropriate, we will omit the bar in the notation $\overline{W}_{x, y}$ and $\overline{W}_{\mathbf{x}_{\Lambda}, \mathbf{y}_{\Lambda}}$ for $x = y$ or $\mathbf{x}_{\Lambda} = \mathbf{y}_{\Lambda}$:

$$\overline{W}_{x, x} = W_{x, x} \quad \text{and} \quad \overline{W}_{\mathbf{x}_{\Lambda}, \mathbf{x}_{\Lambda}} = W_{\mathbf{x}_{\Lambda}, \mathbf{x}_{\Lambda}}.$$

Recall, we refer to $\boldsymbol{\omega}_{\Lambda}$ as a loop configuration and $\overline{\boldsymbol{\omega}}_{\Lambda}$ as a path configuration in Λ . Next, we set:

$$\overline{W} = \bigcup_{x, y \in M} \overline{W}_{x, y} \quad \text{and} \quad \overline{W}_{\Lambda} = \bigcup_{\mathbf{x}_{\Lambda}, \mathbf{y}_{\Lambda} \in M^{\Lambda}} \overline{W}_{\mathbf{x}_{\Lambda}, \mathbf{y}_{\Lambda}}. \quad (2.2.4)$$

2.3. The FK-DLR equations in a finite volume. The aforementioned probability distribution μ_Λ , on space W_Λ , is absolutely continuous relative to the underlying product-measure $\nu_\Lambda (= \nu_{\beta, \Lambda})$, where

$$\begin{aligned} d\nu_\Lambda(\mathbf{x}_\Lambda, \boldsymbol{\omega}_\Lambda) &= \times_{j \in \Lambda} (v(dx(j)) \times P_{x(j), x(j)}(d\omega_j)) \\ &= d\mathbf{x}_\Lambda \times P_{\mathbf{x}_\Lambda, \mathbf{x}_\Lambda}(d\boldsymbol{\omega}_\Lambda). \end{aligned} \quad (2.3.1)$$

Here the measure $P_{x(j), x(j)}(d\omega_j)$ is defined as a Brownian bridge on manifold M with the starting and end point $x(j)$. Next, the Radon–Nikodym derivative (probability density function) $p_\Lambda(\mathbf{x}_\Lambda, \boldsymbol{\omega}_\Lambda) := \frac{d\mu_\Lambda(\mathbf{x}_\Lambda, \boldsymbol{\omega}_\Lambda)}{d\nu_\Lambda(\mathbf{x}_\Lambda, \boldsymbol{\omega}_\Lambda)}$ is of the form

$$p_\Lambda(\mathbf{x}_\Lambda, \boldsymbol{\omega}_\Lambda) = \frac{1}{\Xi_\Lambda} \exp \left[-h^\Lambda(\boldsymbol{\omega}_\Lambda) \right] \quad (2.3.2)$$

where functional $h^\Lambda(\boldsymbol{\omega}_\Lambda)$ has been defined in (2.1.4)–(2.1.5). It is convenient to treat μ_Λ as a Gibbs probability measure for a ‘classical’ spin system where ‘spins’ are represented by loops affiliated with sites $j \in \Lambda$.

To shorten the notation we will omit henceforce the argument \mathbf{x}_Λ and similar arguments from symbols like $p_\Lambda(\mathbf{x}_\Lambda, \boldsymbol{\omega}_\Lambda)$, $d\nu_\Lambda(\mathbf{x}_\Lambda, \boldsymbol{\omega}_\Lambda)$ and $d\mu_\Lambda(\mathbf{x}_\Lambda, \boldsymbol{\omega}_\Lambda)$, bearing in mind that the initial/end-point configuration \mathbf{x}_Λ can be reconstructed from the loop configuration $\boldsymbol{\omega}_\Lambda$.

Measure μ_Λ defines a random field over Λ with realizations $\boldsymbol{\omega}_\Lambda = \{\omega_j, j \in \Lambda\} \in W_\Lambda$ and has the following properties (I), (II).

(I) μ_Λ satisfies the DLR equation over Λ ; cf. Eqn (2.3.4) below. (Recall, $\Lambda \subset \Gamma$ is a finite set.) This means the following. Given $\Lambda^0 \subset \Lambda$, let us agree to write $\boldsymbol{\omega}^0$ for the loop configuration $\boldsymbol{\omega}_{\Lambda^0} \in W_{\Lambda^0}$. Consider the partially integrated probability density

$$p_\Lambda^{\Lambda^0}(\boldsymbol{\omega}^0) := \int_{W_{\Lambda \setminus \Lambda^0}} d\nu_{\Lambda \setminus \Lambda^0}(\boldsymbol{\omega}_{\Lambda \setminus \Lambda^0}) p_\Lambda(\boldsymbol{\omega}^0 \vee \boldsymbol{\omega}_{\Lambda \setminus \Lambda^0}) \quad (2.3.3)$$

where $\boldsymbol{\omega}^0 \vee \boldsymbol{\omega}_{\Lambda \setminus \Lambda^0}$ stands for the concatenation of the two loop configurations yielding a loop configuration over the whole of Λ . Cf. Eqn (2.2.2).

Then, \forall set Λ' such that $\Lambda^0 \subset \Lambda' \subset \Lambda$, the density $p_\Lambda^{\Lambda^0}(\boldsymbol{\omega}^0)$ obeys

$$\begin{aligned} p_\Lambda^{\Lambda^0}(\boldsymbol{\omega}^0) &= \int_{W_{\Lambda \setminus \Lambda'}} d\nu_{\Lambda \setminus \Lambda'}(\boldsymbol{\omega}_{\Lambda \setminus \Lambda'}) \\ &\quad \times p_\Lambda^{\Lambda \setminus \Lambda'}(\boldsymbol{\omega}_{\Lambda \setminus \Lambda'}) \frac{\Xi_{\Lambda' \setminus \Lambda^0}(\boldsymbol{\omega}^0, \boldsymbol{\omega}_{\Lambda \setminus \Lambda'})}{\Xi_{\Lambda'}(\boldsymbol{\omega}_{\Lambda \setminus \Lambda'})}. \end{aligned} \quad (2.3.4)$$

Here $p_{\Lambda}^{\Lambda \setminus \Lambda'}(\omega_{\Lambda \setminus \Lambda'})$ is the partially integrated density similar to (2.3.3):

$$p_{\Lambda}^{\Lambda \setminus \Lambda'}(\omega_{\Lambda \setminus \Lambda'}) := \int_{W_{\Lambda'}} d\nu_{\Lambda'}(\tilde{\omega}_{\Lambda'}) p_{\Lambda}(\tilde{\omega}_{\Lambda'} \vee \omega_{\Lambda \setminus \Lambda'}). \quad (2.3.5)$$

Further, the quantities $\Xi_{\Lambda' \setminus \Lambda^0}(\omega^0, \omega_{\Lambda \setminus \Lambda'})$ and $\Xi_{\Lambda'}(\omega_{\Lambda \setminus \Lambda'})$ are given by the following integrals

$$\begin{aligned} \Xi_{\Lambda' \setminus \Lambda^0}(\omega^0, \omega_{\Lambda \setminus \Lambda'}) &= \int_{W_{\Lambda' \setminus \Lambda^0}} d\nu_{\Lambda' \setminus \Lambda^0}(\omega_{\Lambda' \setminus \Lambda^0}) \\ &\times \exp \left[-h^{\Lambda'}(\omega^0 \vee \omega_{\Lambda' \setminus \Lambda^0} | \omega_{\Lambda \setminus \Lambda'}) \right] \end{aligned} \quad (2.3.6)$$

and

$$\Xi_{\Lambda'}(\omega_{\Lambda \setminus \Lambda'}) = \int_{W_{\Lambda'}} d\nu_{\Lambda'}(\omega_{\Lambda'}) \exp \left[-h^{\Lambda'}(\omega_{\Lambda'} | \omega_{\Lambda \setminus \Lambda'}) \right]. \quad (2.3.7)$$

Next, given loop configurations $\omega_{\Lambda'} = \{\omega_j, j \in \Lambda'\} \in W_{\Lambda'}$ and $\omega_{\Lambda \setminus \Lambda'} = \{\omega_j, j \in \Lambda \setminus \Lambda'\} \in M^{\Lambda \setminus \Lambda'}$, the functional $h^{\Lambda'}(\omega_{\Lambda'} | \omega_{\Lambda \setminus \Lambda'})$ in (2.3.7) is determined by

$$h^{\Lambda'}(\omega_{\Lambda'} | \omega_{\Lambda \setminus \Lambda'}) = h^{\Lambda'}(\omega_{\Lambda'}) + h(\omega_{\Lambda'} | \omega_{\Lambda \setminus \Lambda'}) \quad (2.3.8)$$

where the summand $h^{\Lambda'}(\omega_{\Lambda'})$ is defined as in (2.1.4) and $h(\omega_{\Lambda'} | \omega_{\Lambda \setminus \Lambda'})$ is given by

$$h^{\Lambda'}(\omega_{\Lambda'} | \omega_{\Lambda \setminus \Lambda'}) = \sum_{(j,j') \in \Lambda' \times \Lambda \setminus \Lambda'} h^{j,j'}(\omega_j, \omega_{j'}) \quad (2.3.9)$$

with $h^{j,j'}(\omega_j, \omega_{j'})$ as in (2.1.5).

Finally, the functional $h^{\Lambda'}(\omega^0 \vee \omega_{\Lambda' \setminus \Lambda^0} | \omega_{\Lambda \setminus \Lambda'})$ figuring in (2.3.6), for $\omega^0 \in W_{\Lambda^0}$, $\omega_{\Lambda'} \in W_{\Lambda'}$ and $\omega_{\Lambda \setminus \Lambda'} \in W_{\Lambda \setminus \Lambda'}$, is defined by similar formulas. We say that $h^{\Lambda'}(\omega_{\Lambda'} | \omega_{\Lambda \setminus \Lambda'})$ and $h^{\Lambda'}(\omega^0 \vee \omega_{\Lambda' \setminus \Lambda^0} | \omega_{\Lambda \setminus \Lambda'})$ give the values of a ‘potential energy’ of the loop configurations $\omega_{\Lambda'}$ and $\omega^0 \vee \omega_{\Lambda'}$ in the external field generated by $\omega_{\Lambda \setminus \Lambda'}$. In this context, $\Xi_{\Lambda'}(\omega_{\Lambda \setminus \Lambda'})$ gives the partition function for loop configurations over the ‘intermediate volume’ Λ' in an external potential field generated by the boundary condition $\omega_{\Lambda \setminus \Lambda'} \in W_{\Lambda \setminus \Lambda'}$. Similarly, $\Xi_{\Lambda' \setminus \Lambda^0}(\omega^0, \omega_{\Lambda \setminus \Lambda'})$ can be considered as the partition function in the ‘layer’ $\Lambda' \setminus \Lambda^0$, with an ‘external’ boundary condition $\omega_{\Lambda \setminus \Lambda'} \in W_{\Lambda \setminus \Lambda'}$ and an ‘internal’ loop configuration $\omega^0 \in W_{\Lambda^0}$ (note that ω^0 enters the integral $\Xi_{\Lambda' \setminus \Lambda^0}(\omega^0, \omega_{\Lambda \setminus \Lambda'})$ with its energy $h^{\Lambda^0}(\omega^0)$). A straightforward fact is that

$$\int_{W_{\Lambda^0}} d\nu_{\Lambda^0}(\omega^0) \Xi_{\Lambda' \setminus \Lambda^0}(\omega^0, \omega_{\Lambda \setminus \Lambda'}) = \Xi_{\Lambda'}(\omega_{\Lambda \setminus \Lambda'}).$$

In probabilistic terms, the DLR equation (2.3.4) is equivalent to the following property. Consider the conditional distribution $d\mu_{\Lambda}^{\Lambda^0|\Lambda\setminus\Lambda'}(\omega^0|\omega_{\Lambda\setminus\Lambda'})$ induced by the probability measure μ_{Λ} , for a loop configuration ω^0 over Λ^0 , conditioned by a loop configuration $\omega_{\Lambda\setminus\Lambda'}$ over $\Lambda\setminus\Lambda'$. It is determined by the conditional probability density $p_{\Lambda}^{\Lambda^0|\Lambda\setminus\Lambda'}(\omega^0|\omega_{\Lambda\setminus\Lambda'}) := \frac{d\mu_{\Lambda}^{\Lambda^0|\Lambda\setminus\Lambda'}(\omega^0|\omega_{\Lambda\setminus\Lambda'})}{d\nu_{\Lambda^0}(\omega^0)}$. The equivalent form of the DLR property means that this density has the form

$$p_{\Lambda}^{\Lambda^0|\Lambda\setminus\Lambda'}(\omega^0|\omega_{\Lambda\setminus\Lambda'}) = \frac{\Xi_{\Lambda'\setminus\Lambda^0}(\omega^0, \omega_{\Lambda\setminus\Lambda'})}{\Xi_{\Lambda'}(\omega_{\Lambda\setminus\Lambda'})}. \quad (2.3.10)$$

In fact, μ_{Λ} is the only measure that satisfies the equations (2.3.5), (2.3.10). The name DLR (Dobrushin–Lanford–Ruelle) is widely used in the classical statistical mechanics; see, e.g., [5].

(II) *The measure μ_{Λ} determines the RDMK $F_{\Lambda}^{\Lambda^0}(\mathbf{x}^0, \mathbf{y}^0)$.* Given $\Lambda^0 \subset \Lambda$ and particle configurations $\mathbf{x}^0, \mathbf{y}^0 \in M^{\Lambda^0}$, the RDMK $F_{\Lambda}^{\Lambda^0}(\mathbf{x}^0, \mathbf{y}^0)$ is defined by

$$F_{\Lambda}^{\Lambda^0}(\mathbf{x}^0, \mathbf{y}^0) = \int_{\overline{W}_{\mathbf{x}^0, \mathbf{y}^0}} \mathbb{P}_{\mathbf{x}^0, \mathbf{y}^0}(d\overline{\omega}^0) q_{\Lambda}^{\Lambda^0}(\overline{\omega}^0). \quad (2.3.11)$$

In turn, the functional $q_{\Lambda}^{\Lambda^0}(\overline{\omega}^0)$ is determined by the formula analogous to (2.3.4): $\forall \Lambda'$ such that $\Lambda^0 \subset \Lambda' \subset \Lambda$,

$$\begin{aligned} q_{\Lambda}^{\Lambda^0}(\overline{\omega}^0) &= \int_{W_{\Lambda\setminus\Lambda'}} d\nu_{\Lambda\setminus\Lambda'}(\omega_{\Lambda\setminus\Lambda'}) \\ &\times p_{\Lambda}^{\Lambda\setminus\Lambda'}(\omega_{\Lambda\setminus\Lambda'}) \frac{\Xi_{\Lambda'\setminus\Lambda^0}(\overline{\omega}^0, \omega_{\Lambda\setminus\Lambda'})}{\Xi_{\Lambda'}(\omega_{\Lambda\setminus\Lambda'})} \end{aligned} \quad (2.3.12)$$

with quantities $p_{\Lambda}^{\Lambda\setminus\Lambda'}(\omega_{\Lambda\setminus\Lambda'})$, $\Xi_{\Lambda'\setminus\Lambda^0}(\overline{\omega}^0, \omega_{\Lambda\setminus\Lambda'})$ and $\Xi_{\Lambda^0}(\overline{\omega}_{\Lambda\setminus\Lambda^0})$ defined as in Eqns (2.3.5)–(2.3.8). (The only difference is that a loop configuration ω^0 in (2.3.7) has been replaced with a more general path configuration $\overline{\omega}^0$.) Kernel $F_{\Lambda}^{\Lambda^0}(\mathbf{x}^0, \mathbf{y}^0)$ is often referred to as the reduced DM kernel, briefly RDMK (more precisely, the kernel of the DM in volume Λ , reduced to Λ^0). Accordingly, the functional $q_{\Lambda}^{\Lambda^0}(\overline{\omega}^0)$ can be called a reduced DM functional, briefly RD MF, for $\overline{\omega}^0 \in \overline{W}_{\mathbf{x}^0, \mathbf{y}^0}$. Similarly to (2.3.4), it will be convenient to

write

$$q_{\Lambda}^{\Lambda^0}(\bar{\omega}^0) = \int_{W_{\Lambda \setminus \Lambda'}} d\nu_{\Lambda \setminus \Lambda'}(\omega_{\Lambda \setminus \Lambda'}) \times p_{\Lambda}^{\Lambda \setminus \Lambda'}(\omega_{\Lambda \setminus \Lambda'}) q_{\Lambda}^{\Lambda^0 | \Lambda \setminus \Lambda'}(\bar{\omega}^0 | \omega_{\Lambda \setminus \Lambda'}) \quad (2.3.13)$$

where

$$q_{\Lambda}^{\Lambda^0 | \Lambda \setminus \Lambda'}(\bar{\omega}^0 | \omega_{\Lambda \setminus \Lambda'}) = \frac{\Xi_{\Lambda' \setminus \Lambda^0}(\bar{\omega}^0, \omega_{\Lambda \setminus \Lambda'})}{\Xi_{\Lambda'}(\omega_{\Lambda \setminus \Lambda'})}. \quad (2.3.14)$$

In analogy with Eqn (2.3.10), quantity $q_{\Lambda}^{\Lambda^0 | \Lambda \setminus \Lambda'}(\bar{\omega}^0 | \omega_{\Lambda \setminus \Lambda'})$ in (2.3.14) can be called a conditional RDMF for a path configuration $\bar{\omega}^0 \in \bar{W}_{\mathbf{x}^0, \mathbf{y}^0}$, given a loop configuration $\omega_{\Lambda \setminus \Lambda'} \in W_{\Lambda \setminus \Lambda'}$.

Note that the RDMF $q_{\Lambda}^{\Lambda^0}(\bar{\omega}^0)$ from (2.3.12), (2.3.13) satisfies the invariance relation

$$q_{\Lambda}^{\Lambda^0}(\bar{\omega}^0) = q_{\Lambda}^{\Lambda^0}(g\bar{\omega}^0), \quad g \in \mathbb{G}, \quad \bar{\omega}^0 \in \bar{W}_{\mathbf{x}^0, \mathbf{y}^0} \quad (2.3.15)$$

where

$$g\bar{\omega}^0 = \{g\bar{\omega}_j, j \in \Lambda^0\}, \text{ with } (g\omega_j)(\tau) = g(\omega_j(\tau)), 0 \leq \tau \leq \beta. \quad (2.3.16)$$

Consequently, for RDMK $F_{\Lambda}^{\Lambda^0}(\mathbf{x}^0, \mathbf{y}^0)$ (see (2.3.11)), we have that

$$F_{\Lambda}^{\Lambda^0}(\mathbf{x}^0, \mathbf{y}^0) = F_{\Lambda}^{\Lambda^0}(g\mathbf{x}^0, g\mathbf{y}^0), \quad g \in \mathbb{G}, \quad \mathbf{x}^0, \mathbf{y}^0 \in M^{\Lambda^0} \quad (2.3.17)$$

and for the RDM $R_{\Lambda}^{\Lambda^0}$

$$R_{\Lambda}^{\Lambda^0} = U_{\Lambda^0}(g)^{-1} R_{\Lambda}^{\Lambda^0} U_{\Lambda^0}(g), \quad g \in \mathbb{G}. \quad (2.3.18)$$

However, we will need to consider a similar RDMF $q_{\Lambda | \bar{\mathbf{x}}_{\Gamma' \setminus \Lambda}}^{\Lambda^0}(\bar{\omega}^0)$ defined via operator $\exp [-\beta H_{\Lambda | \bar{\mathbf{x}}_{\Gamma' \setminus \Lambda}}]$ instead of $\exp [-\beta H_{\Lambda}]$. (It can be called a conditional RDMF, with the boundary condition $\bar{\mathbf{x}}_{\Gamma' \setminus \Lambda}$.) The invariance equation

$$q_{\Lambda | \bar{\mathbf{x}}_{\Gamma' \setminus \Lambda}}^{\Lambda^0}(\bar{\omega}^0) = q_{\Lambda | \bar{\mathbf{x}}_{\Gamma' \setminus \Lambda}}^{\Lambda^0}(g\bar{\omega}^0) \quad (2.3.19)$$

(i.e., an analog of (2.3.15)) fails; this makes the statements of Theorems 1.1 and 1.2 non-trivial. (Of course, the covariance property

$$q_{\Lambda | \bar{\mathbf{x}}_{\Gamma' \setminus \Lambda}}^{\Lambda^0}(\bar{\omega}^0) = q_{\Lambda | g\bar{\mathbf{x}}_{\Gamma' \setminus \Lambda}}^{\Lambda^0}(g\bar{\omega}^0)$$

holds true but is useless for our purpose.)

3. The class \mathfrak{G} of Gibbs states in the infinite volume

3.1. Definition of the class \mathfrak{G} . The aim of this section is to define the invariance property (2.3.17) (and consequently, property (2.3.18)) for functionals $q_\Gamma^{\Lambda^0}(\omega^0)$ (and related objects $F_\Gamma^{\Lambda^0}(\mathbf{x}^0, \mathbf{y}^0)$ and $R_\Gamma^{\Lambda^0}$) for the system in an ‘infinite volume’ (i.e., over the whole graph (Γ, \mathcal{E})). That is, we want to prove that functional $q_\Gamma^{\Lambda^0}(\omega^0)$, which we call infinite-volume RDMF, obeys

$$q_\Gamma^{\Lambda^0}(\bar{\omega}^0) = q_\Gamma^{\Lambda^0}(\mathbf{g}\bar{\omega}^0), \quad \mathbf{g} \in \mathbf{G}, \quad \bar{\omega}^0 \in \bar{W}_{\Lambda^0}. \quad (3.1.1)$$

The formal definition of infinite-volume RDMF $q_\Gamma^{\Lambda^0}(\omega^0)$ ($q^{\Lambda^0}(\omega^0)$ for short) related to the system over (Γ, \mathcal{E}) requires additional constructions and will lead us to the definition of the aforementioned class of states \mathfrak{G} ; see below. At this point we state that the key step is to establish an asymptotical form of (2.3.19) for infinite-volume conditional RDMF $q^{\Lambda^0|\Gamma\setminus\Lambda'}(\bar{\omega}^0|\omega_{\Gamma\setminus\Lambda'})$ when set Λ' is ‘large enough’. In essence, we will prove that, \forall finite set $\Lambda^0 \subset \Gamma$,

$$\lim_{n \rightarrow \infty} \frac{q^{\Lambda^0|\Gamma\setminus\Lambda(n)}(\mathbf{g}\bar{\omega}^0|\omega_{\Gamma\setminus\Lambda(n)})}{q^{\Lambda^0|\Gamma\setminus\Lambda(n)}(\bar{\omega}^0|\omega_{\Gamma\setminus\Lambda(n)})} = 1, \quad (3.1.2)$$

uniformly in: (i) group element $\mathbf{g} \in \mathbf{G}$, (ii) path configuration $\bar{\omega} \in \bar{W}_{\Lambda^0}$, (iii) an (infinite) loop configuration $\omega_{\Gamma\setminus\Lambda(n)} \in W_{\Gamma\setminus\Lambda(n)}$ representing an infinite-volume external boundary condition. Here and below $\Lambda(n) = \Lambda(o, n)$ and $\Sigma(n)$ mean the ball and the sphere of radius n (cf. (1.1.4)) around a reference point $o \in \Gamma$ (the choice of point o will not matter).

In fact, functional $q^{\Lambda^0|\Gamma\setminus\Lambda(n)}(\bar{\omega}^0|\omega_{\Gamma\setminus\Lambda(n)})$ is itself defined as the limit

$$q^{\Lambda^0|\Gamma\setminus\Lambda(n)}(\bar{\omega}^0|\omega_{\Gamma\setminus\Lambda(n)}) = \lim_{r \rightarrow \infty} q_{\Lambda(r)}^{\Lambda^0|\Lambda(r)\setminus\Lambda(n)}(\bar{\omega}^0|\omega_{\Lambda(r)\setminus\Lambda(n)}) \quad (3.1.3)$$

where, for $\Lambda^0 \subset \Lambda(n) \subset \Lambda(r)$, the value $q_{\Lambda(r)}^{\Lambda^0|\Lambda(r)\setminus\Lambda(n)}(\bar{\omega}^0|\omega_{\Lambda(r)\setminus\Lambda(n)})$ has been determined in Eqn (2.3.14).

At this point it is appropriate to establish some probabilistic background. The Borel sigma-algebra of subsets of the loop space W is denoted by \mathfrak{W} . Given a finite subset $\Lambda \subset \Gamma$, we obtain the induced sigma-algebra of subsets of W_Λ which is denoted by \mathfrak{W}_Λ . Similarly, for a trajectory space \bar{W} the sigma-algebra $\bar{\mathfrak{W}}$ is defined which leads to the sigma-algebra $\bar{\mathfrak{W}}_{\Lambda^0}$ of subsets in \bar{W}_{Λ^0} . For $\Lambda' \subset \Lambda$, the sigma-algebra $\mathfrak{W}_{\Lambda'}$ is naturally identified with a

sub-sigma-algebra of \mathfrak{W}_Λ which is denoted by the same symbol $\mathfrak{W}_{\Lambda'}$). For the whole graph Γ , we can introduce the Cartesian product W_Γ considered as the countable set of loop configurations $\{\omega_j, j \in \Gamma\}$, $\omega_j \in M$; as earlier, a loop ω_j is associated with site $j \in \Gamma$. For a finite set $\Lambda \subset \Gamma$, the sigma-algebra \mathfrak{W}_Λ can again be identified with the sigma-algebra of subsets of W_Γ ; as before, it is convenient to use the same notation for both. The sigma-algebra \mathfrak{W}_Γ is defined as the smallest sigma-algebra of subsets of W_Γ containing $\mathfrak{W}_\Lambda \forall$ finite $\Lambda \subset \Gamma$. In a similar fashion we define the sigma-algebra $\mathfrak{W}_{\Gamma \setminus \Lambda^0} \subset \mathfrak{W}_\Gamma$ for a given (finite) set $\Lambda^0 \subset \Gamma$; as before, it is naturally identified with the sigma-algebra of subsets in $W_{\Gamma \setminus \Lambda^0}$.

Let us now define the class \mathfrak{G} of states of the C^* -algebra \mathfrak{B} . As before, a state φ of \mathfrak{B} is identified with a family of RDMs $R^{\Lambda^0} = R_\varphi^{\Lambda^0}$ where Λ^0 is an arbitrary finite subset of Γ ; each R^{Λ^0} is a positive definite operator in \mathcal{H}_{Λ^0} of trace one, and the compatibility relation (1.4.9) holds true. In short, for a state $\varphi \in \mathfrak{G}$ the RDMs R^{Λ^0} are integral operators (see (3.1.4)), with integral kernels $F^{\Lambda^0}(\mathbf{x}^0, \mathbf{y}^0)$ satisfying (3.1.5)–(3.1.12), where the probability measure μ_Γ obeys (3.1.13)–(3.1.17). Properties (3.1.5)–(3.1.17) are direct analogs of the corresponding properties of the RDMs $F_\Lambda^{\Lambda^0}(\mathbf{x}^0, \mathbf{y}^0)$ and $F_{\Lambda|\mathbf{x}_{\Gamma' \setminus \Lambda}}^{\Lambda^0}(\mathbf{x}^0, \mathbf{y}^0)$ in a finite volume Λ .

Passing to the formal presentation, the RDM R^{Λ^0} is determined by its integral kernel $F^{\Lambda^0}(\mathbf{x}^0, \mathbf{y}^0)$:

$$(R^{\Lambda^0} \phi)(\mathbf{x}^0) = \int_{M^{\Lambda^0}} d\mathbf{y}^0 F^{\Lambda^0}(\mathbf{x}^0, \mathbf{y}^0) \phi(\mathbf{y}^0), \quad \mathbf{x}^0 \in M^{\Lambda^0}. \quad (3.1.4)$$

In turn, the RDMK $F^{\Lambda^0}(\mathbf{x}^0, \mathbf{y}^0) = f_\varphi^{\Lambda^0}(\mathbf{x}^0, \mathbf{y}^0)$ is obtained via a functional $q^{\Lambda^0}(\overline{\omega}^0) = q_{\varphi, \Gamma}^{\Lambda^0}(\overline{\omega}^0)$ referred to as an infinite-volume RDMF:

$$F^{\Lambda^0}(\mathbf{x}^0, \mathbf{y}^0) = \int_{\overline{W}_{\mathbf{x}^0, \mathbf{y}^0}} P_{\mathbf{x}^0, \mathbf{y}^0}^\beta(d\overline{\omega}^0) q^{\Lambda^0}(\overline{\omega}^0). \quad (3.1.5)$$

Further, the infinite-volume RDMF for a state φ under consideration, should admit a particular representation. Namely, there exists a probability measure $\mu_\Gamma = \mu_{\varphi, \Gamma}$ on $(W_\Gamma, \mathfrak{W}_\Gamma)$ such that \forall finite set $\Lambda' \subset \Gamma$ with $\Lambda^0 \subset \Lambda'$,

$$q^{\Lambda^0}(\overline{\omega}^0) = \int_{W_{\Gamma \setminus \Lambda'}} d\mu_\Gamma^{\Gamma \setminus \Lambda'}(\omega_{\Gamma \setminus \Lambda'}) q^{\Lambda^0|\Gamma \setminus \Lambda'}(\overline{\omega}^0 | \omega_{\Gamma \setminus \Lambda'}) \quad (3.1.6)$$

where

$$q^{\Lambda^0|\Gamma\setminus\Lambda'}(\bar{\omega}^0|\omega_{\Gamma\setminus\Lambda'}) = \frac{\Xi_{\Lambda'\setminus\Lambda^0}(\bar{\omega}^0, \omega_{\Gamma\setminus\Lambda'})}{\Xi_{\Lambda'}(\omega_{\Gamma\setminus\Lambda'})}. \quad (3.1.7)$$

Here $\mu_{\Gamma}^{\Gamma\setminus\Lambda'}$ stands for the restriction of measure μ_{Γ} to the sigma-algebra $\mathfrak{W}_{\Gamma\setminus\Lambda'}$.

Moreover, the expressions $\Xi_{\Lambda'\setminus\Lambda^0}(\bar{\omega}^0, \omega_{\Gamma\setminus\Lambda'})$ and $\Xi_{\Lambda'}(\omega_{\Gamma\setminus\Lambda'})$ represent, as before, partition functions in $\Lambda'\setminus\Lambda^0$ and Λ' , with the corresponding boundary conditions:

$$\begin{aligned} \Xi_{\Lambda'\setminus\Lambda^0}(\bar{\omega}^0, \omega_{\Gamma\setminus\Lambda'}) &= \int_{W_{\Lambda'\setminus\Lambda^0}} d\nu_{\Lambda'\setminus\Lambda^0}(\omega_{\Lambda'\setminus\Lambda^0}) \\ &\times \exp \left[-h^{\Lambda'}(\bar{\omega}^0 \vee \omega_{\Lambda'\setminus\Lambda^0} | \omega_{\Gamma\setminus\Lambda'}) \right] \end{aligned} \quad (3.1.8)$$

and

$$\Xi_{\Lambda'}(\omega_{\Gamma\setminus\Lambda'}) = \int_{W_{\Lambda'}} d\nu_{\Lambda'}(\omega_{\Lambda'}) \exp \left[-h^{\Lambda'}(\omega_{\Lambda'} | \omega_{\Gamma\setminus\Lambda'}) \right]. \quad (3.1.9)$$

The functional $h^{\Lambda'}$ is defined by formulas similar to (2.3.8), (2.3.9):

$$h^{\Lambda'}(\omega_{\Lambda'} | \omega_{\Gamma\setminus\Lambda'}) = h^{\Lambda'}(\omega_{\Lambda'}) + h(\omega_{\Lambda'} | \omega_{\Gamma\setminus\Lambda'}) \quad (3.1.10)$$

and

$$\begin{aligned} h^{\Lambda'}(\bar{\omega}^0 \vee \omega_{\Lambda'\setminus\Lambda^0} | \omega_{\Gamma\setminus\Lambda'}) &= h^{\Lambda'}(\bar{\omega}^0 \vee \omega_{\Lambda'\setminus\Lambda^0}) \\ &+ h(\bar{\omega}^0 \vee \omega_{\Lambda'\setminus\Lambda^0} | \omega_{\Gamma\setminus\Lambda'}) \end{aligned} \quad (3.1.11)$$

where

$$h(\omega_{\Lambda'} | \omega_{\Gamma\setminus\Lambda'}) = \sum_{(j,j') \in \Lambda' \times (\Gamma \setminus \Lambda')} h^{j,j'}(\omega_j, \omega_{j'}) \quad (3.1.12)$$

and similarly for $h^{\Lambda'}(\bar{\omega}^0 \vee \omega_{\Lambda'\setminus\Lambda^0} | \omega_{\Gamma\setminus\Lambda'})$. In turn, the terms $h^{\Lambda'}$ and $h^{j,j'}$ are as in (2.3.8), (2.3.9). It is assumed that the series in (3.1.12) is convergent for $\mu_{\Gamma}^{\Gamma\setminus\Lambda'}$ -almost all $\omega_{\Gamma\setminus\Lambda'} \in W_{\Gamma\setminus\Lambda'}$.

The functionals $h^{\Lambda'}(\omega_{\Lambda'})$, $h^{\Lambda'}(\omega_{\Lambda'} | \omega_{\Gamma\setminus\Lambda'})$, $h^{\Lambda'}(\bar{\omega}^0 \vee \omega_{\Lambda'\setminus\Lambda^0})$, $h^{\Lambda'}(\bar{\omega}^0 \vee \omega_{\Lambda'\setminus\Lambda^0} | \omega_{\Gamma\setminus\Lambda'})$ and $h^{\Lambda'}(\bar{\omega}^0 \vee \omega_{\Lambda'\setminus\Lambda^0} | \omega_{\Gamma\setminus\Lambda'})$ have the same meaning in terms of ‘energies’ of loop/path configurations as before.

The measure μ_{Γ} figuring in Eqns (3.1.6) and (3.1.8) has to satisfy the infinite-volume DLR equations similar to (2.3.5). Namely, consider $p_{\Gamma}^{\Lambda^0}(\omega^0)$,

the probability density function, relative to $d\nu^{\Lambda^0}(\omega^0)$, for the measure $\mu_\Gamma^{\Lambda^0}$, the restriction to the sigma-algebra $\mathfrak{M}(\Lambda^0)$ of measure μ_Γ :

$$p_\Gamma^{\Lambda^0}(\omega^0) = \frac{d\mu_\Gamma^{\Lambda^0}(\omega^0)}{d\nu^{\Lambda^0}(\omega^0)}, \quad \omega^0 \in W_{\Lambda^0} \quad (3.1.13)$$

The equations for μ_Γ are that \forall finite sets Λ^0 and Λ' where $\Lambda^0 \subset \Lambda' \subset \Gamma$,

$$p_\Gamma^{\Lambda^0}(\omega^0) = \int_{W_{\Gamma \setminus \Lambda'}} d\mu_\Gamma^{\Gamma \setminus \Lambda'}(\omega_{\Gamma \setminus \Lambda'}) p_\Gamma^{\Lambda^0 | \Gamma \setminus \Lambda'}(\omega^0 | \omega_{\Gamma \setminus \Lambda'}). \quad (3.1.14)$$

Here $p_\Gamma^{\Lambda^0 | \Gamma \setminus \Lambda'}(\omega^0 | \omega_{\Gamma \setminus \Lambda'})$ is the conditional probability density for ω^0 , conditioned by boundary condition $\omega_{\Gamma \setminus \Lambda'} \in W_{\Gamma \setminus \Lambda'}$:

$$p_\Gamma^{\Lambda^0 | \Gamma \setminus \Lambda'}(\omega^0 | \omega_{\Gamma \setminus \Lambda'}) = \frac{\Xi_{\Lambda' \setminus \Lambda^0}(\omega^0, \omega_{\Gamma \setminus \Lambda^0})}{\Xi_{\Lambda'}(\omega_{\Gamma \setminus \Lambda'})}. \quad (3.1.15)$$

Here, as in (3.1.8)

$$\begin{aligned} \Xi_{\Lambda' \setminus \Lambda^0}(\omega^0, \omega_{\Gamma \setminus \Lambda^0}) &= \int_{W_{\Lambda' \setminus \Lambda^0}} d\nu_{\Lambda' \setminus \Lambda^0}(\omega_{\Lambda' \setminus \Lambda^0}) \\ &\times \exp \left[-h^{\Lambda'}(\omega^0 \vee \omega_{\Lambda' \setminus \Lambda^0} | \omega_{\Gamma \setminus \Lambda'}) \right] \end{aligned} \quad (3.1.16)$$

and, as in (3.1.9),

$$\Xi_{\Lambda'}(\omega_{\Gamma \setminus \Lambda'}) = \int_{W_{\Lambda'}} d\nu_{\Lambda'}(\omega_{\Lambda'}) \exp \left[-h^{\Lambda'}(\omega_{\Lambda'} | \omega_{\Gamma \setminus \Lambda'}) \right], \quad (3.1.17)$$

with the functional $h^{\Lambda'}$ defined similarly to Eqns (3.1.10)–(3.1.12).

Remark 3.1. We do not claim (at least in this paper and its sequel [8]) that the properties (3.1.4)–(3.1.17) imply that the operator R^{Λ^0} is an RDM (positive definiteness of R^{Λ^0} remains an open question). However, when one can assert (on grounds of some additional information) that a given family of operator $\{R^{\Lambda^0}\}$ obeying (3.1.4)–(3.1.17) consists of positive-definite operators then we can speak of a state $\varphi \in \mathfrak{G}$. (Viz., this is the case of limiting Gibbs states discussed in Theorems 1.1 and 1.2.) As it stands, our results stated in Section 3.2 hold true for any family of operators R^{Λ^0} for which Eqns (3.1.4)–(3.1.17) are fulfilled. E.g., we can claim the assertion of Theorem 1.2 for

any linear normalized functional on \mathfrak{B} defined by a family $\{R^{\Lambda^0}\}$ satisfying (3.1.4)–(3.1.17).

Some elements of the above construction have been used in the literature; see, e.g., [12] and references therein.

We will refer to Eqns (3.1.6)–(3.1.12) as an FK-DLR representation (of the infinite-volume RDMF $q^{\Lambda^0}(\omega^0)$) by a given probability measure μ_Γ , assuming that μ_Γ satisfies the infinite-volume DLR equations (3.1.14). It is important to stress that, unlike the case of a finite $\Lambda \subset \Gamma$, the solution to the infinite-volume FK-DLR equations (3.1.14) over the whole graph Γ may be, in general, non-unique. However, the family of functionals q^{Λ^0} , where Λ^0 runs through the finite subsets of Γ is determined uniquely provided that a measure μ_Γ is given, satisfies (3.1.6), (3.1.7). In accordance with the above scheme, this gives rise to the family of RDMs F^Λ and – ultimately – RDMs R^Λ , for finite sets $\Lambda \subset \Gamma$, obeying the above compatibility property (1.4.9). The corresponding state (emerging from the probability measure μ_Γ) is denoted by $\varphi_\Gamma (= \varphi(\mu_\Gamma))$; when possible, the subscript Γ will be omitted. Given $\beta \in (0, \infty)$, the class of the measures $\mu_\Gamma = \mu_{\beta, \Gamma}$ satisfying Eqns (3.1.13) is denoted by $\mathfrak{G}(\beta)$, as well as the class of related states φ_Γ .

In Theorem 3.1 below we establish that the class $\mathfrak{G}(\beta)$ is non-empty \forall given $\beta \in (0, \infty)$.

As was said earlier, the infinite-volume invariance property under study is expressed by Eqn (3.1.1): $\forall \mathbf{g} \in \mathbf{G}$, finite set $\Lambda^0 \subset \Gamma$, $\mathbf{x}^0, \mathbf{y}^0 \in M^{\Lambda^0}$ and $\overline{\omega}^0 = \{\overline{\omega}_j, j \in \Lambda^0\} \in \overline{W}_{\mathbf{x}^0, \mathbf{y}^0}$, the value $q_\beta^{\Lambda^0}(\mathbf{g}\overline{\omega}^0) = q_\beta^{\Lambda^0}(\overline{\omega}^0)$. Here $\mathbf{g}\overline{\omega}^0$ is as in (2.3.16). A similar property for the density $p^{\Lambda^0}(\overline{\omega}^0)$ has the form: \forall finite set $\Lambda^0 \subset \Gamma$ and loop configuration $\overline{\omega}^0 = \{\overline{\omega}_j, j \in \Lambda^0\} \in \overline{W}_{\Lambda^0}$,

$$p^{\Lambda^0}(\overline{\omega}^0) = p^{\Lambda^0}(\mathbf{g}\overline{\omega}^0), \quad \mathbf{g} \in \mathbf{G}. \quad (3.1.18)$$

The invariance properties in Eqns (3.1.1) and (3.1.18) imply that, \forall finite set $\Lambda^0 \subset \Gamma$, the infinite-volume RDMs $R_\Gamma^{\Lambda^0}$ have the property similar to (1.4.9):

$$R_\Gamma^{\Lambda^0} U_{\Lambda^0}(\mathbf{g}) = U_{\Lambda^0}(\mathbf{g}) R_\Gamma^{\Lambda^0} \quad (3.1.19)$$

which, in terms of the corresponding state φ , means (1.4.10).

3.2. Properties of class \mathfrak{G} . The results of this paper about state class \mathfrak{G} are summarised in Theorems 3.1–3.4 below. In Theorems 3.1–3.3 we assume the above conditions (1.1.1), (1.1.2), (1.3.2)–(1.3.5).

Theorem 3.1. *For all $\beta \in (0, \infty)$, the sequence of Gibbs states $\varphi_{\Lambda(n)}$ contains a subsequence $\varphi_{\Lambda(n_k)}$ such that \forall finite $\Lambda^0 \subset \Gamma$ and $A_0 \in \mathfrak{B}_{\Lambda^0}$, we have:*

$$\lim_{k \rightarrow \infty} \varphi_{\Lambda(n_k)}(A_0) = \varphi(A_0)$$

where state $\varphi \in \mathfrak{G}(\beta)$. Consequently, class $\mathfrak{G}(\beta)$ is non-empty. \triangleleft

Theorem 3.2. *For all $\beta \in (0, \infty)$ and finite $\Lambda^0 \subset \Gamma$, any Gibbs state $\varphi \in \mathfrak{G}(\beta)$ satisfies properties (3.1.1) and (3.1.18)–(3.1.19). \triangleleft*

The invariance property can be formally extended to ground states. We call a state $\overline{\varphi}$ (of C^* -algebra \mathfrak{B}) a ground state if there exists a sequence of states $\varphi_n \in \mathfrak{G}(\beta_n)$ with $\beta_n \rightarrow \infty$ such that $\overline{\varphi} = w^* - \lim_{n \rightarrow \infty} \varphi_n$.

Corollary 3.3. *Any ground state $\overline{\varphi}$ (i.e., a w^* -limiting point of states $\varphi_n \in \mathfrak{G}(\beta_n)$) obeys (3.1.1) and (3.1.18)–(3.1.19). \triangleleft*

Of course, Corollary 3.3 does not prove existence of ground states for the model under consideration.

In a future paper we will remove the smoothness condition upon the potential function V (see (1.3.2)), by following the methodology from [16]. In this paper we note that, like the classical case (cf. [7] and the bibliography therein), if the condition of smoothness is violated, the symmetry property may be destroyed. See Theorem 3.4 below.

Theorem 3.4. *Take $\Gamma = \mathbb{Z}^2$, the regular square lattice, with distance $d(j, j') = \max [|j_1 - j'_1|, |j_2 - j'_2|]$. Take $M = S^1 = \mathbb{G}$ where $S^1 = \mathbb{R}/\mathbb{Z}$ is a unit circle, with a standard metric $\rho(x, x') = \min [|x - x'|, 1 - |x - x'|]$ and the group operation of addition mod 1. Assume that the two-body potentials $J(d(j, j'))$ and $V(x, x')$, $j, j' \in \mathbb{Z}^2$, $x, x' \in S^1$, are of the form*

$$\begin{aligned} J(d(j, j')) &= \begin{cases} 1, & |j - j'| = 1, \\ 0, & |j - j'| \neq 1, \end{cases} \\ V(x, x') &= \begin{cases} -\cos 2\pi(x - x'), & \rho(x, x') \leq \theta, \\ +\infty, & \rho(x, x') > \theta, \end{cases} \end{aligned} \tag{3.2.1}$$

with a usual agreement $0 \cdot (+\infty) = 0$, where $\theta \in (0, 1/4)$ is a constant. In this case, Hamiltonian H_Λ is equipped with Dirichlet's boundary conditions on

$D \subset M^\Lambda$ where

$$D = \left\{ \mathbf{x}_\Lambda \in M^\Lambda : |x(j) - x(j')| \geq \theta \text{ for some } j, j' \in \Lambda \text{ with } \mathbf{d}(j, j') = 1 \right\}.$$

Then, $\forall \beta \in (0, \infty)$, there exists an FK-DLR measure $\tilde{\mu} = \tilde{\mu}_\beta \in \mathfrak{G}$ which is not S^1 -invariant. Consequently, the corresponding FK-DLR state $\tilde{\varphi} = \tilde{\varphi}_{\tilde{\mu}} \in \mathfrak{G}(\beta)$ is not S^1 -invariant. \triangleleft

Similarly to a ground state $\overline{\varphi}$, we can define $\overline{\mu}$, a ground-state FK-measure. Namely, take an FK-DLR measure $\mu_n \in \mathfrak{G}(\beta_n)$ and consider its image $\hat{\mu}_n$ under projection $\omega_\Gamma \in W_\Gamma^\beta \mapsto \mathbf{x}_\Gamma \in M^\Gamma$ where $\mathbf{x}_\Gamma = \mathbf{x}_\Gamma(\omega_\Gamma)$ is the collection of initial points for loop configuration ω_Γ . Suppose that $\hat{\mu}$ is a limiting point for sequence $\hat{\mu}_n$ as $\beta_n \rightarrow \infty$. Then we say that $\hat{\mu}$ is a ground-state FK-measure. Furthermore, such a measure is called \mathbf{G} -invariant if \forall finite set $\Lambda^0 \subset \Gamma$, $\mathbf{g} \in \mathbf{G}$ and a (bounded) function $\phi : M^{\Lambda^0} \rightarrow \mathbb{R}$, the integral $\hat{\mu}(U_{\Lambda^0}(\mathbf{g})\phi) = \hat{\mu}(\phi)$.

Corollary 3.5. *Suppose that the family of non- S^1 -invariant FK-DLR measures $\tilde{\mu}_\beta$ specified in Theorem 3.4 has a limiting point $\tilde{\psi}$ as $\beta \rightarrow \infty$. Then $\tilde{\psi}$ is a non- S^1 -invariant ground-state FK-measure. \triangleleft*

4. Proof of the main results

In this section we deliver proofs of the stated results.

4.1. Proof of Theorem 3.1. Given finite sets Λ^0 and Λ , $\Lambda^0 \subset \Lambda \subset \Gamma$, the RDMK $F_{\Lambda}^{\Lambda^0}$ (see (2.2.1)) is a continuous function on $M^{\Lambda^0} \times M^{\Lambda^0}$. The first observation is that the sequence $F_{\Lambda(n)}^{\Lambda^0}$ is compact in $C^0(M^{\Lambda^0} \times M^{\Lambda^0})$ by the Ascoli–Arzela theorem, as these functions are uniformly bounded and continuous. The latter property is based upon conditions (1.3.2)–(1.3.4) and the assumption that M is compact.

More precisely, to show uniform boundedness, note that \forall finite $\Lambda' \subset \Gamma$ with $\Lambda' \supset \Lambda^0$, the conditional RDMF $q^{\Lambda^0|\Gamma \setminus \Lambda'}$ (cf. Eqn (3.1.7)) satisfies

$$q^{\Lambda^0|\Gamma \setminus \Lambda'}(\bar{\omega}^0 | \omega_{\Gamma \setminus \Lambda'}) \leq (e^{2\beta \bar{J}V})^{\sharp \Lambda^0} \quad (4.1.1)$$

for all path configurations $\bar{\omega}^0 \in \bar{W}_{\Lambda^0}$, $\omega_{\Gamma \setminus \Lambda'} \in W_{\Gamma \setminus \Lambda'}$. Here, the constant \bar{J} is given by

$$\bar{J} = \sup \left[\sum_{j' \in \Gamma} J(d(j, j')) : j \in \Gamma \right] \quad (4.1.2)$$

where J is the function from (1.3.3). (In fact, \bar{J} coincides with the quantity $\bar{J}(1)$ in (1.3.1).) A similar bound holds if we replace Γ with ball $\Lambda(n) \subset \Gamma \setminus \Lambda'$. After integration, this yields the estimate

$$F_{\Lambda(n)}^{\Lambda^0}(\mathbf{x}^0, \mathbf{y}^0) \leq (e^{2\beta \bar{J}V} \bar{p}^{\beta})^{\sharp \Lambda^0} \quad (4.1.3)$$

where

$$\bar{p}^{\beta} = \sup [p^{\beta}(x, y), |\nabla_x p^{\beta}(x, y)|, |\nabla_y p^{\beta}(x, y)| : x, y \in M]. \quad (4.1.4)$$

Here $p^{\beta}(x, y)$ stands for the transition probability density from x to y for the Brownian motion (with the generator $-\Delta/2$ where Δ is the Laplace operator on M) in time β :

$$p^{\beta}(x, y) = \frac{1}{(2\pi\beta)^{d/2}} \sum_{\underline{n}=(n_1, \dots, n_d) \in \mathbb{Z}^d} \exp \left(-|x - y + \underline{n}|^2/2\beta \right), \quad (4.1.5)$$

The argument for uniform continuity (or equi-continuity) of RDMKs $F_{\Lambda(n)}^{\Lambda^0}(\mathbf{x}^0, \mathbf{y}^0)$ is more technical. We want to check that the gradients

$\nabla_{\mathbf{x}^0} F_{\Lambda(n)}^{\Lambda^0}(\mathbf{x}^0, \mathbf{y}^0)$ and $\nabla_{\mathbf{y}^0} F_{\Lambda(n)}^{\Lambda^0}(\mathbf{x}^0, \mathbf{y}^0)$ are uniformly bounded. There are two contributions into the gradient: one comes from varying the measure $\mathbb{P}_{\mathbf{x}^0, \mathbf{y}^0}(d\bar{\omega}^0)$, the other from varying the functional $\exp[-h^{\Lambda^0}(\bar{\omega}^0 | \omega_{\Lambda(n) \setminus \Lambda^0})]$. The first contribution can be uniformly bounded in terms of the constant \bar{p}^β .

The second contribution can be analysed by deforming a path $\bar{\omega}_j \in \bar{\omega}^0$, $j \in \Lambda^0$: one of the end-points $x(j)$ or $y(j)$ can be moved, say, along a geodesic on M (i.e., a straight line). The points on the path are then moved, at a scaled distance, via a parallel transfer (the affine connection on the torus is trivial). This contribution is related to the differentiation of the exponent and controlled due to the bound (1.3.2) on the derivatives of the potential.

The second contribution yields an expression of the form

$$\int_{\bar{W}_{\mathbf{x}^0, \mathbf{y}^0}} \mathbb{P}_{\mathbf{x}^0, \mathbf{y}^0}(d\bar{\omega}^0) \sum_{j \in \Lambda^0} \tilde{h}_j(\omega(j), \omega_{\Lambda(n) \setminus \Lambda^0}) \times \exp \left[-h^{\Lambda^0}(\bar{\omega}^0 | \omega_{\Lambda(n) \setminus \Lambda^0}) \right] \quad (4.1.6)$$

where functional $\tilde{h}_j(\bar{\omega}^0, \omega_{\Lambda(n) \setminus \Lambda^0})$ is uniformly bounded. Combining this with the argument used to estimate the RDMK $F_{\Lambda(n)}^{\Lambda^0}(\mathbf{x}^0, \mathbf{y}^0)$ allows one to bound the gradients $\nabla_{\mathbf{x}^0} F_{\Lambda(n)}^{\Lambda^0}(\mathbf{x}^0, \mathbf{y}^0)$ and $\nabla_{\mathbf{y}^0} F_{\Lambda(n)}^{\Lambda^0}(\mathbf{x}^0, \mathbf{y}^0)$ as well.

More precisely, given a site $j \in \Lambda^0$, we need to differentiate in $\underline{x}(j)$ or $\underline{y}(j)$ the expression

$$\begin{aligned} & \sum_{\mathbf{n}} \int_{\bar{W}_{\mathbf{x}^0, \mathbf{y}^0 + \mathbf{n}}} \mathbb{P}_{\mathbf{x}^0, \mathbf{y}^0 + \mathbf{n}}(d\bar{\omega}) \exp[-h^{\Lambda^0}(\bar{\omega} | \omega_{\Lambda(n) \setminus \Lambda^0})] \\ &= \sum_{\mathbf{n}} \exp \left(-|\mathbf{x}^0 - \mathbf{y}^0 - \mathbf{n}|^2 / 2\beta \right) \\ & \quad \times \int_{W_{\mathbf{x}^0, \mathbf{x}^0}} \mathbb{P}_{\mathbf{x}^0, \mathbf{x}^0}(d\bar{\omega}^0) \exp \left[-h^{\Lambda^0}(\bar{\omega}^0 + \zeta_{\mathbf{n}} | \omega_{\Lambda(n) \setminus \Lambda^0}) \right]. \end{aligned}$$

Here we sum over vectors $\mathbf{n} = (\underline{n}(j), j \in \Lambda^0) \in (\mathbb{Z}^d)^{\Lambda^0}$. Furthermore, $\zeta_{\mathbf{n}}$ is a linear map:

$$\zeta_{\mathbf{n}}(\tau) = \frac{\tau}{\beta}(\mathbf{y}^0 + \mathbf{n} - \mathbf{x}^0), \quad 0 \leq \tau \leq \beta.$$

Finally, the measures $\mathbb{P}_{\mathbf{x}^0, \mathbf{y}^0 + \mathbf{n}}$ and $\mathbb{P}_{\mathbf{x}^0, \mathbf{x}^0}$ refer to the standard Brownian motion on \mathbb{R}^d .

Suppose we differentiate in $y(j) \in \mathbb{R}^d$. Differentiating the exponent yields a convergent series. Next,

$$\begin{aligned} & \nabla_{y(j)} \exp \left[-h^{\Lambda^0}(\bar{\omega}^0 + \zeta_{\mathbf{n}}|\omega_{\Lambda(n) \setminus \Lambda^0}) \right] \\ &= \left(\sum_{l \in \Lambda(n) \setminus \Lambda^0} \int_0^\beta d\tau \frac{\tau}{\beta} (\nabla_{y(j)} V)(\omega(j, \tau), \omega(l, \tau)) \right) \\ & \quad \times \exp \left[-h^{\Lambda^0}(\bar{\omega}^0 + \zeta_{\mathbf{n}}|\omega_{\Lambda(n) \setminus \Lambda^0}) \right] \end{aligned}$$

is bounded due to (1.3.2). (The expression in the big brackets gives the term $\tilde{h}_j(\omega(j), \omega_{\Lambda(n) \setminus \Lambda^0})$ figuring in Eqn (4.1.6).) This yields the desired result.

Differentiation in $x(j)$ can be done in a similar manner, by exchanging \mathbf{x}^0 and \mathbf{y}^0 in the above series.

Now let an RDMK F^{Λ^0} be a limiting point for $F_{\Lambda(n_k)}^{\Lambda^0}$ in $C^0(M^{\Lambda^0} \times M^{\Lambda^0})$. Then we have that

$$\lim_{k \rightarrow \infty} \int_{M^{\Lambda} \times M^{\Lambda}} d\mathbf{x}^0 d\mathbf{y}^0 \left[F_{\Lambda(n_k)}^{\Lambda^0}(\mathbf{x}^0, \mathbf{y}^0) - F^{\Lambda^0}(\mathbf{x}^0, \mathbf{y}^0) \right]^2 = 0.$$

In other words, the RDM $R_{\Lambda(n_k)}^{\Lambda^0}$ in \mathcal{H}^{Λ^0} converges to the infinite-volume RDM R^{Λ^0} in the Hilbert-Schmidt norm: $\|R_{\Lambda(n_k)}^{\Lambda^0} - R^{\Lambda^0}\|_{\text{HS}} \rightarrow 0$. According to Lemma 1 from [23], the convergence takes place in the trace-norm as well: $\|R_{\Lambda(n_k)}^{\Lambda^0} - R^{\Lambda^0}\|_{\text{tr}} \rightarrow 0$. We obtain that the sequence of states $\varphi_{\Lambda(n)}$ is w^* -compact.

In parallel, an argument can be developed that the measures μ_{Λ} form a compact family as $\Lambda \nearrow \Gamma$. More precisely, we would like to show that \forall finite $\Lambda^0 \subset \Gamma$, the family of measures $\mu_{\Lambda}^{\Lambda^0}$ is compact. To see this, it suffices to check that, for a fixed Λ^0 , the sequence $\{\mu_{\Lambda(n)}^{\Lambda(L)}, n = L+1, L+2, \dots\}$ is tight and apply the Prokhorov theorem.

To check tightness, we use the two facts. (i) The reference measure $d\nu_{\Lambda^0}$ on W^{Λ^0} (see Eqn (2.3.1)) is supported by loop configurations with the standard continuity modulus $\sqrt{2\epsilon \ln(1/\epsilon)}$. (ii) The probability density

$p_{\Lambda}^{\Lambda^0}(\omega^0) = \frac{d\mu_{\Lambda(n)}^{\Lambda(L)}(\Omega(\Lambda(L)))}{d\nu_{\Lambda(n)}(\Omega(\Lambda(L)))}$ (cf. Eqn (2.3.3)) is bounded from above by a constant $\exp \{ \beta [\sharp \Lambda(L)] \bar{V} \bar{J}^* \}$ (and from below by $\exp \{ -\beta [\sharp \Lambda(L)] \bar{V} \bar{J}^* \}$). See (1.3.2) and (1.3.4).

The constructed family of limit-point measures μ^{Λ^0} has the compatibility property and therefore satisfies the assumptions of the Kolmogorov theorem. The result is that there exists a unique probability measure μ on W^Γ such that the restriction of μ on the sigma-algebra of subsets localized in Λ^0 coincides with μ^{Λ^0} .

The fact that μ is FK-DLR follows from the above construction. Hence, each limit point φ falls in class $\mathfrak{G}(\beta)$. \square

Remark 4.1. Anticipating a forthcoming result for a general compact manifold M , we propose to discuss a version of the above argument for an example where M is a two-dimensional Klein bottle with a flat Riemannian metric. A convenient representation is through the universal simply connected cover which in this case is the Euclidean plane \mathbb{R}^2 with the standard metric. For the fundamental polygon we take a square $[-1/2, 1/2] \times [-1/2, 1/2]$ where the following pairs of points $(x_1; x_2)$ are glued:

$$(x_1; -1/2) \text{ and } (-x_1; 1/2), \text{ where } -1/2 < x_1 < 1/2$$

and

$$(-1/2; x_2) \text{ and } (1/2; x_2), \text{ where } -1/2 < x_2 < 1/2.$$

The cover map $T : \mathbb{R}^2 \rightarrow M$ is as follows:

$$\begin{aligned} T(x_1, x_2) &\mapsto ((-1)^{n_2}(x_1 - n_1); x_2 - n_2) \\ \text{whenever } &-1/2 \leq x_i - n_i < 1/2, \quad -1/2 \leq x_2 - n_2 < 1/2. \end{aligned} \quad (4.1.7)$$

Here and below, $n_1, n_2 \in \mathbb{Z}$ are integers. (In this example, \mathbb{G} may be a circle \mathbb{S}^1 (a 1D torus) realized as an interval $[-1/2, 1/2]$ with points $-1/2$ and $1/2$ glued together. The action is: $gx = (x_1 + g, x_2)$, for $x = (x_1; x_2)$, with addition in $[-1/2, 1/2]$. (Other choices of \mathbb{G} (a 1D torus of length 2 or a 2D torus) are analysed in a similar fashion.)

In this example, the integral

$$\int_{\overline{W}_{\mathbf{x}^0, \mathbf{y}^0}} \mathbb{P}_{\mathbf{x}^0, \mathbf{y}^0}(d\overline{\omega}^0) \exp \left[-h^{\Lambda^0}(\overline{\omega}^0 | \omega_{\Lambda(n) \setminus \Lambda^0}) \right] \quad (4.1.8)$$

contributing to $F_{\Lambda}^{\Lambda^0}(\mathbf{x}^0, \mathbf{y}^0)$ can again be differentiated explicitly. For definiteness, take Λ^0 to be a one-point set, say $\{o\}$, with particle configurations \mathbf{x}^0 and \mathbf{y}^0 reduced to single points in M (or rather in the fundamental polygon):

$$\mathbf{x}^0 = x = (x_1; x_2), \quad \mathbf{y}^0 = y = (y_1; y_2), \quad -1/2 \leq x_i, y_i \leq 1/2, \quad i = 1, 2.$$

Then the above integral takes the form

$$\begin{aligned} & \frac{1}{2\pi\beta} \sum_{n_1, n_2 \in \mathbb{Z}} \exp \left[-((-1)^{n_2} x_1 - y_1 - n_1)^2 / (2\beta) \right] \\ & \quad \times \exp \left[-(x_2 - y_2 - n_2)^2 / (2\beta) \right] \int_{W_{x,x}^{\beta, \mathbb{R}^2}} \mathbb{P}_{x,x}^{\beta, \mathbb{R}^2} (d\omega_1^0 \times d\omega_2^0) \quad (4.1.9) \\ & \quad \times \exp \left[-h^{\Lambda^0} \left(T \left[\left(\omega_1^0 + \delta_1^{(n_1)}, \omega_2^0 + \delta_2^{(n_2)} \right) \right] \middle| \omega_{\Lambda(n) \setminus \Lambda^0} \right) \right]. \end{aligned}$$

Here $W_{x,x}^{\beta, \mathbb{R}^2}$ and $\mathbb{P}_{x,x}^{\beta, \mathbb{R}^2}$ stand, respectively, for the space of continuous loops (closed trajectories beginning and ending at x) and the Brownian bridge distribution in the plane \mathbb{R}^2 of the time-length β . Next, $\delta_i^{(n_i)}$ are functions $[0, \beta] \rightarrow \mathbb{R}$ providing deformations of plane loops ω_i^0 from $W_{x_i, x_i}^{\beta, \mathbb{R}^2}$ into plane paths $\omega_1^0 + \delta_1^{(n_1)}$ from $\overline{W}_{x_i, y_i + n_i}^{\beta, \mathbb{R}^2}$:

$$\delta_1^{(n_1)}(\tau) = \frac{\tau}{\beta} (y_1 - x_1 + n_1), \quad \delta_1^{(n_2)}(\tau) = \frac{\tau}{\beta} (y_2 - x_2 + n_2). \quad (4.1.10)$$

A similar formula holds after replacing $W_{x,x}^{\beta, \mathbb{R}^2}$ and $\mathbb{P}_{x,x}^{\beta, \mathbb{R}^2}$ with $W_{y,y}^{\beta, \mathbb{R}^2}$ and $\mathbb{P}_{y,y}^{\beta, \mathbb{R}^2}$, *mutatis mutandis*. Differentiations ∇_x and ∇_y then become straightforward (although rather tedious), confirming the above claim about uniform continuity of RDMs $F_\Lambda^{\Lambda^0}$ (cf. Eqn (4.1.6)).

Theorem 1.1 can be deduced from Theorem 3.1 since every limiting Gibbs state $\varphi \in \mathfrak{G}^0$ lies in \mathfrak{G} (in other words, $\mathfrak{G}^0 \subseteq \mathfrak{G}$), as follows from the above argument and the observations made in Sections 2.3 and 3.1.

4.2. Proof of Theorem 3.2. We follow the approach initiated in [4]. The proof of Theorem 3.2 is based on the following bound for infinite-volume RDMFs: \forall finite $\Lambda^0 \subset \Gamma$, $\overline{\omega}^0 \in \overline{W}_\Lambda$ and $\mathbf{g} \in \mathbb{G}$,

$$q^{\Lambda^0}(\mathbf{g}\overline{\omega}^0) + q^{\Lambda^0}(\mathbf{g}^{-1}\overline{\omega}^0) \geq 2q^{\Lambda^0}(\overline{\omega}^0). \quad (4.2.1)$$

Lemma 1 from [4] implies that (3.1.1) follows from (4.2.1).

In turn, the bound (4.2.1) follows from a similar inequality for the conditional RDMFs: \forall finite $\Lambda^0 \subset \Gamma$, $\overline{\omega}^0 \in \overline{W}_\Lambda$, $\mathbf{g} \in \mathbb{G}$ and $a \in (1, \infty)$, for any n large enough and $\omega_{\Gamma \setminus \Lambda(n)}$,

$$\begin{aligned} & aq^{\Lambda^0 | \Gamma \setminus \Lambda(n)}(\mathbf{g}\overline{\omega}^0 | \omega_{\Gamma \setminus \Lambda(n)}) + aq^{\Lambda^0 | \Gamma \setminus \Lambda(n)}(\mathbf{g}^{-1}\overline{\omega}^0 | \omega_{\Gamma \setminus \Lambda(n)}) \\ & \geq 2q^{\Lambda^0 | \Gamma \setminus \Lambda(n)}(\overline{\omega}^0 | \omega_{\Gamma \setminus \Lambda(n)}). \end{aligned} \quad (4.2.2)$$

In fact, to deduce (4.2.1) from (4.2.2), it is enough to integrate (4.2.2) in $d\mu_{\Gamma}^{\Gamma \setminus \Lambda(n)}(\omega_{\Gamma \setminus \Lambda(n)})$ and let $a \searrow 1$.

Now, Eqn (4.2.2) is deduced after performing a special construction related to a family of ‘gauge’ actions $\mathbf{g}_{\Lambda(n) \setminus \Lambda^0}$ on loop configurations $\omega_{\Lambda(n) \setminus \Lambda^0}$; see Eqns (4.2.4), (4.2.5) below. A particular feature of action $\mathbf{g}_{\Lambda(n) \setminus \Lambda^0}$ is that it ‘decays’ to \mathbf{e} , the unit element of \mathbf{G} (which generates a ‘trivial’ identity action), when we move from Λ^0 towards $\Gamma \setminus \Lambda(n)$. Formally, (4.2.2) will follow from the inequality: \forall finite $\Lambda^0 \subset \Gamma$, $\overline{\omega}^0 \in \overline{W}_{\Lambda}$, $\mathbf{g} \in \mathbf{G}$ and $a \in (1, \infty)$, for any n large enough, $\omega_{\Lambda(n) \setminus \Lambda^0}$ and $\omega_{\Gamma \setminus \Lambda(n)}$,

$$\begin{aligned} & \frac{a}{2} \exp \left[-h^{\Lambda(n)}((\mathbf{g}\overline{\omega}^0) \vee (\mathbf{g}_{\Lambda(n) \setminus \Lambda^0} \omega_{\Lambda(n) \setminus \Lambda^0}) | \omega_{\Gamma \setminus \Lambda(n)}) \right] \\ & + \frac{a}{2} \exp \left[-h^{\Lambda(n)}((\mathbf{g}^{-1}\overline{\omega}^0) \vee (\mathbf{g}_{\Lambda(n) \setminus \Lambda^0}^{-1} \omega_{\Lambda(n) \setminus \Lambda^0}) | \omega_{\Gamma \setminus \Lambda(n)}) \right] \\ & \geq \exp \left[-h^{\Lambda(n)}(\overline{\omega}^0 \vee \omega_{\Lambda(n) \setminus \Lambda^0} | \omega_{\Gamma \setminus \Lambda(n)}) \right]. \end{aligned} \quad (4.2.3)$$

Indeed, (4.2.2) follows from (4.2.3) by integrating in $d\nu_{\Lambda(n) \setminus \Lambda^0}(\omega_{\Lambda(n) \setminus \Lambda^0})$ and normalizing by $\Xi_{\Lambda(n) \setminus \Lambda^0}(\omega_{\Gamma \setminus \Lambda(n)})$; cf. Eqn (3.1.16) with $\Lambda' = \Lambda(n)$. (Here one uses the fact that the Jacobian of the map $\omega_{\Lambda(n) \setminus \Lambda^0} \mapsto \mathbf{g}_{\Lambda(n) \setminus \Lambda^0} \omega_{\Lambda(n) \setminus \Lambda^0}$ equals 1.)

Thus, our aim becomes to prove (4.2.3). The gauge family $\mathbf{g}_{\Lambda(n) \setminus \Lambda^0}$ is composed by individual actions $\mathbf{g}_j^{(n)} \in \mathbf{G}$:

$$\mathbf{g}_{\Lambda(n) \setminus \Lambda^0} = \{\mathbf{g}_j^{(n)}, j \in \Lambda(n) \setminus \Lambda^0\}. \quad (4.2.4)$$

Let us identify the element $\mathbf{g} \in \mathbf{G}$ with a vector $\underline{\theta} = \theta A \in M$ and use the additive notation: $\mathbf{g}x := x + \underline{\theta}$, $x \in M$. Then $\mathbf{g}_j^{(n)} \in \mathbf{G}$ corresponds to multiples of the vector $\underline{\theta}$. Namely, we fix a positive integer value \bar{r} such that $\Lambda^0 \subset \Lambda_{\bar{r}}$ and identify

$$\mathbf{g}_j^{(n)} \text{ with } \underline{\theta}v(n, j) \quad (4.2.5)$$

where

$$v(n, j) = \begin{cases} 1, & d(o, j) \leq \bar{r}, \\ \vartheta(d(j, o) - \bar{r}, n - \bar{r}), & d(o, j) > \bar{r}. \end{cases} \quad (4.2.6)$$

In turn, the function $\vartheta(a, b)$ satisfies

$$\vartheta(a, b) = \mathbf{1}(a \leq 0) + \frac{\mathbf{1}(0 < a < b)}{Q(b)} \int_a^b z(u) du, \quad a, b \in \mathbb{R}, \quad (4.2.7)$$

with the same functions $Q(b)$ and $z(u)$ as proposed in [4]

$$Q(b) = \int_0^b z(u) du, \quad (4.2.8)$$

$$\text{where } z(u) = \mathbf{1}(u \leq 2) + \mathbf{1}(u > 2) \frac{1}{u \ln u}, \quad b > 0.$$

Moreover, $\mathbf{g}_{\Lambda(n) \setminus \Lambda^0}^{-1}$ is the collection of the inverse elements:

$$\mathbf{g}_{\Lambda(n) \setminus \Lambda^0}^{-1} = \left\{ \mathbf{g}_j^{(n)^{-1}}, j \in \Lambda(n) \setminus \Lambda^0 \right\}.$$

We will use the formulas for $\mathbf{g}_j^{(n)}$ for $j \in \Lambda(n)$, or even for $j \in \Gamma$, as they agreed with the requirement that $\mathbf{g}_j^{(n)} \equiv \mathbf{g}$ when $j \in \Lambda^0$ and $\mathbf{g}_j^{(n)} \equiv \mathbf{e}$ for $j \in \Gamma \setminus \Lambda(n)$. Accordingly, we will use the notation $\mathbf{g}_{\Lambda(n)} = \{\mathbf{g}_j^{(n)}, j \in \Lambda(n)\}$.

Next, we use the invariance property (1.3.5). The Taylor formula for function $V \in C^2$ yields for $j, j' \in \Lambda(n)$:

$$\begin{aligned} & \left| V \left(\mathbf{g}_j^{(n)} \omega_j, \mathbf{g}_{j'}^{(n)} \omega_{j'} \right) \right. \\ & \quad \left. + V \left(\mathbf{g}_j^{(n)^{-1}} \omega_j, \mathbf{g}_{j'}^{(n)^{-1}} \omega_{j'} \right) - 2V(\omega_j, \omega_{j'}) \right| \\ & \leq C |\underline{\theta}|^2 |v(n, j) - v(n, j')|^2 \overline{V}. \end{aligned} \quad (4.2.9)$$

Here $C \in (0, \infty)$ is a constant \overline{V} is taken from (1.3.2) and notations from (4.2.5) are used.

The bound (4.2.9) is crucial and exploits the structure of the group action. It uses the fact that the first-order terms in the expansion in the left-hand side of (4.2.9) cancel each other because of the presence of elements $\mathbf{g}_j^{(n)}$ and $\mathbf{g}_{j'}^{(n)}$ and their inverses, $\mathbf{g}_j^{(n)^{-1}}$ and $\mathbf{g}_{j'}^{(n)^{-1}}$. (This idea goes back to [14] and [4].)

The term $|v(n, j) - v(n, j')|^2$ can be specified as

$$|v(n, j) - v(n, j')|^2 = \begin{cases} 0, & \text{if } \mathbf{d}(j, o), \mathbf{d}(j', o) \leq \bar{r}, \\ 0, & \text{if } \mathbf{d}(j, o), \mathbf{d}(j', o) \geq n, \\ \left[\vartheta(\mathbf{d}(j, o) - \bar{r}, n - \bar{r}) - \vartheta(\mathbf{d}(j', o) - \bar{r}, n - \bar{r}) \right]^2, & \text{if } \bar{r} < \mathbf{d}(j, o), \mathbf{d}(j', o) \leq n, \\ \vartheta(\mathbf{d}(j, o) - \bar{r}, n - \bar{r})^2, & \text{if } \bar{r} < \mathbf{d}(j, o) \leq n, \mathbf{d}(j', o) \in]\bar{r}, n[, \\ \vartheta(\mathbf{d}(j', o) - \bar{r}, n - \bar{r})^2, & \text{if } \bar{r} < \mathbf{d}(j', o) \leq n, \mathbf{d}(j, o) \in]\bar{r}, n[. \end{cases} \quad (4.2.10)$$

By using convexity of the function \exp and Eqn (4.2.9), $\forall a > 1$,

$$\begin{aligned} & \frac{a}{2} \exp \left[-h^{\Lambda(n)} \left(\mathbf{g}_{\Lambda(n)}(\bar{\omega}^0 \vee \omega_{\Lambda(n) \setminus \Lambda^0}) | \omega_{\Gamma \setminus \Lambda(n)} \right) \right] \\ & + \frac{a}{2} \exp \left[-h^{\Lambda(n)} \left(\mathbf{g}_{\Lambda(n)}^{-1}(\bar{\omega}^0 \vee \omega_{\Lambda(n) \setminus \Lambda^0}) | \omega_{\Gamma \setminus \Lambda(n)} \right) \right] \\ & \geq a \exp \left[-\frac{1}{2} h^{\Lambda(n)} \left(\mathbf{g}_{\Lambda(n)}(\bar{\omega}^0 \vee \omega_{\Lambda(n) \setminus \Lambda^0}), \omega_{\Gamma \setminus \Lambda(n)} \right) \right. \\ & \quad \left. - \frac{1}{2} h^{\Lambda(n)} \left(\mathbf{g}_{\Lambda(n)}^{-1}(\bar{\omega}^0 \vee \omega_{\Lambda(n) \setminus \Lambda^0}) | \omega_{\Gamma \setminus \Lambda(n)} \right) \right] \\ & \geq a \exp \left[-h^{\Lambda(n)} \left(\bar{\omega}^0 \vee \omega_{\Lambda(n) \setminus \Lambda^0} | \omega_{\Gamma \setminus \Lambda(n)} \right) \right] e^{-C\Psi/2} \end{aligned} \quad (4.2.11)$$

where

$$\Psi = \Psi(n, \mathbf{g}) = |\underline{\theta}|^2 \sum_{(j, j') \in \Lambda(n) \times \Gamma} J(\mathbf{d}(j, j')) |v(n, j) - v(n, j')|^2. \quad (4.2.12)$$

The next remark is that

$$\begin{aligned} \Psi & \leq 3|\underline{\theta}|^2 \sum_{(j, j') \in \Lambda(n) \times \Gamma} \mathbf{1}(\mathbf{d}(j, o) \leq \mathbf{d}(j', o)) J_{j, j'} \\ & \quad \times \left[\vartheta(\mathbf{d}(j, o) - \bar{r}, n - \bar{r}) - \vartheta(\mathbf{d}(j', o) - \bar{r}, n - \bar{r}) \right]^2 \end{aligned} \quad (4.2.13)$$

where, with the help of the triangle inequality, for all $j, j' : \mathbf{d}(j, o) \leq \mathbf{d}(j', o)$

$$\begin{aligned} 0 & \leq \vartheta(\mathbf{d}(j, o) - \bar{r}, n - \bar{r}) - \vartheta(\mathbf{d}(j', o) - \bar{r}, n - \bar{r}) \\ & \leq \mathbf{d}(j, j') \frac{z(\mathbf{d}(j, o) - \bar{r})}{Q(n - \bar{r})}. \end{aligned} \quad (4.2.14)$$

This yields

$$\begin{aligned}\Psi &\leq \frac{3\|\underline{\theta}\|^2}{Q(n-\bar{r})^2} \sum_{(j,j') \in \Lambda(n) \times \Gamma} J(\mathbf{d}(j,j')) \mathbf{d}(j,j')^2 z(\mathbf{d}(j,o) - \bar{r})^2 \\ &\leq \frac{3\|\underline{\theta}\|^2}{Q(n-\bar{r})^2} \left[\sup_{j \in \Gamma} \sum_{j' \in \Gamma} J(\mathbf{d}(j,j')) \mathbf{d}(j,j')^2 \right] \sum_{j \in \Lambda_{n+r_0}} z(\mathbf{d}(j,o) - \bar{r})^2.\end{aligned}$$

In view of (1.3.4) it remains to estimate the sum $\sum_{j \in \Lambda_{n+r_0}} z(\mathbf{d}(j,o) - \bar{r})^2$. To this end, observe that $uz(u) < 1$ when $u \in (3, \infty)$. The next remark is that the number of sites in the sphere Σ_n grows linearly with n . Consequently,

$$\begin{aligned}\sum_{j \in \Lambda(n+r_0)} z(\mathbf{d}(j,o) - \bar{r})^2 &= \sum_{1 \leq k \leq n+r_0} z(k - \bar{r}) \sum_{j \in \Sigma_k} z(k - \bar{r}) \\ &\leq C_0 \sum_{1 \leq k \leq n+r_0} z(k - \bar{r}) \leq C_1 Q(n+r_0 - \bar{r})\end{aligned}$$

and

$$\Psi \leq \frac{C}{Q(n-\bar{r})} \rightarrow \infty, \text{ as } n \rightarrow \infty.$$

Therefore, given $a > 1$ for n large enough, the term $ae^{-C\Psi/2}$ in the RHS of (4.2.11) becomes > 1 . Hence,

$$\begin{aligned}&\frac{a}{2} \exp \left[-h^{\Lambda(n)} \left(\mathbf{g}_{\Lambda(n)}(\bar{\omega}^0 \vee \omega_{\Lambda(n) \setminus \Lambda^0}) | \bar{\omega}_{\Gamma \setminus \Lambda(n)} \right) \right] \\ &+ \frac{a}{2} \exp \left[-h^{\Lambda(n)} \left(\mathbf{g}_{\Lambda(n)}^{-1}(\bar{\omega}^0 \vee \omega_{\Lambda(n) \setminus \Lambda^0}) | \omega_{\Gamma \setminus \Lambda(n)} \right) \right] \\ &\geq \exp \left[-h^{\Lambda(n)} \left(\bar{\omega}^0 \vee \omega_{\Lambda(n) \setminus \Lambda^0} | \omega_{\Gamma \setminus \Lambda(n)} \right) \right].\end{aligned} \tag{4.2.15}$$

Eqn (4.2.15) implies that the conditional RDMF

$$\begin{aligned}q^{\Lambda^0 | \Gamma \setminus \Lambda(n)}(\bar{\omega}^0 | \omega_{\Gamma \setminus \Lambda(n)}) &= \int_{W_{\Lambda(n) \setminus \Lambda^0}} d\nu_{\Lambda(n) \setminus \Lambda^0}(\omega_{\Lambda(n) \setminus \Lambda^0}) \\ &\times \frac{\exp \left[-h^{\Lambda^0}(\bar{\omega}^0 \vee \omega_{\Lambda(n) \setminus \Lambda^0} | \omega_{\Gamma \setminus \Lambda^0}) \right]}{\Xi_{\Lambda(n)}(\omega_{\Gamma \setminus \Lambda(n)}),}\end{aligned} \tag{4.2.16}$$

obeys

$$\begin{aligned}\lim_{n \rightarrow \infty} \left[q^{\Lambda^0 | \Gamma \setminus \Lambda(n)}(\mathbf{g}\bar{\omega}^0 | \omega_{\Gamma \setminus \Lambda(n)}) + q^{\Lambda^0 | \Gamma \setminus \Lambda(n)}(\mathbf{g}^{-1}\bar{\omega}^0 | \omega_{\Gamma \setminus \Lambda(n)}) \right] \\ \geq 2 \lim_{n \rightarrow \infty} q^{\Lambda^0 | \Gamma \setminus \Lambda(n)}(\bar{\omega}^0 | \omega_{\Gamma \setminus \Lambda(n)})\end{aligned} \tag{4.2.17}$$

uniformly in boundary condition $\omega_{\Gamma \setminus \Lambda(n)}$. Integrating (4.2.17) in $d\mu_{\Gamma}^{\Gamma \setminus \Lambda(n)}(\omega_{\Gamma \setminus \Lambda(n)})$ yields (4.2.3). \square

4.3. Proof of Theorem 3.4 and Lemma 1.1. In Theorem 3.4 our argument follows the idea proposed in [7]. On the lattice \mathbb{Z}^2 consider the squares

$$\Lambda(n) = \{j = (j_1, j_2) : \max [|j_1|, |j_2|] \leq n\}, n = 1, 2, \dots$$

The outer boundary of $\Lambda(n)$ is the set

$$\Sigma(n+1) = \{j = (j_1, j_2) : \max [|j_1|, |j_2|] = n+1\}.$$

Fix a point $x^* \in S^1$, a value $\beta \in (0, \infty)$ and consider a state $\varphi^* = \varphi(\mu^*)$ induced by measure $\mu^* = \mu^{(x^*)} \in \mathfrak{G}(\beta)$ which is a limiting point for the family of measures $\mu_n^* = \mu_{\beta, \Lambda(n)}^{(x^*)}$ (cf. (2.3.10)) as $n \rightarrow \infty$. Here μ_n^* stands for the probability distribution on $W_{\Lambda(n)}$ with the ‘cooled’ boundary condition

$$\omega_{\Sigma(n+1)}^* = \{\omega_j^*, j \in \Sigma(n+1)\} \quad (4.3.1)$$

where

$$\omega_j^*(\tau) \equiv x^*, \quad 0 \leq \tau \leq \beta. \quad (4.3.2)$$

Without loss of generality, assume that $\mu^* = \lim_{n \rightarrow \infty} \mu_n^*$. To simplify the notation, let us also omit the subscript $*$, writing $\mu = \mu^*$. If state $\varphi = \varphi^*$ is not S^1 -invariant, we are done. Otherwise, suppose that φ is S^1 -invariant. Then choose an arc $\alpha = (x^* - 1/200, x^* + 1/200)$ of length $1/100$ around the point x^* and let $\Pi_0(\alpha)$ be the orthoprojection on the subspace in $\mathcal{H}_0 \simeq \mathcal{H}$ formed by functions supported by arc α . Then

$$\varphi(\Pi_0(\alpha)) = \int_{W_{\{0\}}} d\mu^{\{0\}}(\omega_0) \mathbf{1}(x_0(\omega_0) \in \alpha) = \frac{1}{100}. \quad (4.3.3)$$

(The lower/upper scripts 0 and $\{0\}$ indicate that we take $\Lambda^0 = \{0\}$, i.e., consider spins attached to lattice site $0 \in \mathbb{Z}^2$.)

Hence, for n large enough, the conditional distribution $d\mu^{(\{0\}|\Sigma(n+1))}(\omega_0 | \omega_{\Sigma(n+1)}^*)$ for $\omega_0 \in W_{\{0\}}$, given boundary condition $\omega_{\Sigma(n+1)}^*$, satisfies:

$$\int_{W_{\{0\}}} d\mu^{(\{0\}|\Sigma(n+1))}(\omega_0 | \omega_{\Sigma(n+1)}^*) \mathbf{1}(x_0(\omega_0) \in \alpha) < \frac{1}{99}. \quad (4.3.4)$$

Next, given $\eta \in (0, 1]$, consider a family of points

$$\tilde{x}_{j,\eta} = x^* + j_1 \eta \theta \bmod 1, \quad j = (j_1, j_2) \in \mathbb{Z}^2,$$

and the family of the corresponding cooled loops $\{\tilde{\omega}_{j,\eta}\}$:

$$\tilde{\omega}_{j,\eta}(\tau) \equiv \tilde{x}_{j,\eta}, \quad 0 \leq \tau \leq \beta. \quad (4.3.5)$$

Further, consider the loop configuration $\tilde{\omega}_{\Sigma(n+1),\eta} = \{\tilde{\omega}_{j,\eta}, j \in \Sigma(n+1)\}$ over the boundary $\Sigma(n+1)$ formed by loops $\tilde{\omega}_{j,\eta}$. For $\eta = 1$, the only configuration over $\Lambda(n)$ compatible with the boundary condition $\tilde{\omega}_{\Sigma(n+1),\eta}$ is the one where all loops coincide with $\tilde{\omega}_{\eta,j}$: for any other choice of the configuration, the energy $h^{\Lambda(n)|\Sigma(n+1)}$ is equal to $+\infty$. By continuity, for each n , there exists $\tilde{\eta}(n) \in (0, 1)$ such that the probability measures $d\mu^{\{\emptyset\}|\Sigma(n+1)}(\omega_0 | \tilde{\omega}_{\Sigma(n+1),\tilde{\eta}(n)})$ conditional on $\tilde{\omega}_{\Sigma(n+1),\tilde{\eta}(n)}$ satisfy

$$\int_{W_{\{\emptyset\}}} d\mu^{\{\emptyset\}|\Sigma(n+1)}(\omega_0 | \tilde{\omega}_{\Sigma(n+1),\tilde{\eta}(n)}) \mathbf{1}(x_0(\omega_0) \in \alpha) = \frac{2}{3}, \quad (4.3.6)$$

and $\tilde{\eta}(n)$ is uniformly separated from 0 and 1. Any limiting point $\tilde{\mu}$ of the sequence of conditional measures $\mu^{\Lambda(n)|\Sigma(n+1)}(\cdot | \tilde{\omega}_{\Sigma(n+1),\tilde{\eta}(n)})$, $n \rightarrow \infty$, yields

$$\int_{W_{\{\emptyset\}}} d\tilde{\mu}^{\{\emptyset\}}(\omega_0) \mathbf{1}(x_0(\omega_0) \in \alpha) = \frac{2}{3}, \quad (4.3.7)$$

and the induced state $\tilde{\varphi}$ gives $\tilde{\varphi}(\Pi_\alpha(0)) = 2/3$. It means that neither $\tilde{\mu}$ nor $\tilde{\varphi}$ are S^1 -invariant. \square

Proof of Corollary 3.5. Eqn (4.3.7) guarantees that $\forall \beta > 0$ there exists a non- S^1 -invariant measure $\tilde{\mu}_\beta \in \mathfrak{G}(\beta)$. Passing to a limiting point as $\beta \rightarrow \infty$ yields a ground-state measure $\tilde{\psi}$ with the property that

$$\int_{W_{\{\emptyset\}}} d\tilde{\psi}^{\{\emptyset\}}(\omega_0) \mathbf{1}(x_0(\omega_0) \in \alpha) = \frac{2}{3}, \quad (4.3.8)$$

again contradicting S^1 -invariance. \square

Proof of Lemma 1.1. Let $\lambda_1 \geq \lambda_2 \geq \dots$ be the sequence of the eigenvalues of operator R and $e_i(x)$, $i = 1, 2, \dots$ be the corresponding eigenvectors. As follows from (1.4.11),

$$\lim_{n \rightarrow \infty} \sum_{i,j} \left(\langle R_n e_i, e_j \rangle - \lambda_i \delta_{ij} \right)^2 = 0.$$

We want to show that the sequence $\{R_n\}$ converges to R in the Banach space \mathfrak{C} of the trace-class operators in $L_2(M, \nu)$ with the norm $\|\cdot\|_1$. We are going to use a natural basis in \mathfrak{C} formed by the system of rank one ‘matrix units’ $E_{ij} = |e_i\rangle\langle e_j|$. Set:

$$R_n^{(i_0)} = \sum_{1 \leq i, j < i_0} \langle R_n e_i, e_j \rangle E_{ij}, \quad \overline{R}_n^{(i_0)} = \sum_{i, j \geq i_0} \langle R_n e_i, e_j \rangle E_{ij},$$

$$\widetilde{R}_n^{(i_0)} = \sum_{1 \leq i < i_0} \sum_{j \geq i_0} \langle R_n e_i, e_j \rangle E_{ij}.$$

Next, set

$$R^{(i_0)} = \sum_{1 \leq i < i_0} \lambda_i E_{ii}.$$

Clearly, $R_n^{(i_0)}$ and $\overline{R}_n^{(i_0)}$ are positive-definite operators. Furthermore,

$$\|R_n^{(i_0)}\|_1 + \|\overline{R}_n^{(i_0)}\|_1 = 1$$

and

$$R_n = R_n^{(i_0)} + \overline{R}_n^{(i_0)} + \widetilde{R}_n^{(i_0)} + \left(\widetilde{R}_n^{(i_0)}\right)^*.$$

Take an arbitrary $\epsilon > 0$ and choose $i_0 = i_0(\epsilon)$ and $n_0 = n_0(\epsilon)$ such that

$$\sum_{i \geq i_0} \lambda_i < \frac{\epsilon}{8}$$

and for $n \geq n_0$

$$\|R_n^{(i_0)} - R^{(i_0)}\|_1 < \frac{\epsilon}{8}, \quad \sum_{i \neq j} (\langle R_n e_i, e_j \rangle)^2 < \frac{\epsilon^2}{\sqrt{2} i_0^2}.$$

Then for $n \geq n_0$,

$$\begin{aligned} \|R - R_n\|_1 &\leq \|R - R^{(i_0)}\|_1 + \|R^{(i_0)} - R_n^{(i_0)}\|_1 + \|R_n^{(i_0)} - R_n\|_1 \\ &\leq \epsilon/8 + \epsilon/8 + \|R_n - R_n^{(i_0)}\|_1. \end{aligned}$$

It remains to estimate the term $\|R_n - R_n^{(i_0)}\|_1$. To this end we write:

$$\begin{aligned} \|R_n - R_n^{(i_0)}\|_1 &\leq \|\overline{R}_n^{(i_0)}\|_1 + 2\|\widetilde{R}_n^{(i_0)}\|_1 = 1 - \|R_n^{(i_0)}\|_1 + 2\|\widetilde{R}_n^{(i_0)}\|_1 \\ &\leq 1 - \|R^{(i_0)}\|_1 + \epsilon/8 + 2\|\widetilde{R}_n^{(i_0)}\|_1 \leq 1 - \epsilon/4 + 2\|\widetilde{R}_n^{(i_0)}\|_1. \end{aligned}$$

Finally,

$$\left\| \tilde{R}_n^{(i_0)} \right\|_1 \leq \sum_{1 \leq i < i_0} \left[\sum_{j \geq i_0} \langle R_n e_i, e_j \rangle^2 \right]^{1/2} < i_0 \left[\sum_{i \neq j} \langle R_n e_i, e_j \rangle^2 \right]^{1/2}$$

which is $< \epsilon/2$. This completes the proof of Lemma 1.1. \square

Acknowledgement. This work has been conducted under Grant 2011/20133-0 provided by the FAPESP, Grant 2011.5.764.45.0 provided by The Reitoria of the Universidade de São Paulo and Grant 2012/04372-7 provided by the FAPESP. The authors express their gratitude to NUMEC and IME, Universidade de São Paulo, Brazil, for the warm hospitality. The authors thank the referees for remarks and suggestions.

References

- [1] S. Albeverio, Y. Kondratiev, Y. Kozitsky, M. Röckner, The Statistical Mechanics of Quantum Lattice Systems. A Path Integral Approach. EMS Publishing House, Zürich, 2009.
- [2] O. Bratteli, D. Robinson. Operator Algebras and Quantum Statistical Mechanics. Vol. I: C^* - and W^* -Algebras. Symmetry Groups. Decomposition of States; Vol. II: Equilibrium States. Models in Quantum Statistical Mechanics. Springer-Verlag, Berlin, 2002.
- [3] R. L. Dobrushin and S. B. Shlosman. Absence of breakdown of continuous symmetry in two-dimensional models of statistical physics. *Commun. Math. Phys.*, **42**, 1975, 30-40
- [4] J. Fröhlich and C. Pfister. On the absence of spontaneous symmetry breaking and of crystalline ordering in two-dimensional systems. *Commun. Math. Phys.*, **81**, 1981, 277–298
- [5] H.-O. Georgii, Gibbs Measures and Phase Transitions. Walter de Gruyter, Berlin, 1988
- [6] J. Ginibre. Some applications of functional integration in statistical mechanics. In: *Statistical mechanics and quantum field theory* (C.M. DeWitt, R. Stora (eds)). Gordon and Breach, 1973, pp. 327- 428.
- [7] D. Ioffe, S. Shlosman, Y. Velenik. 2D models of statistical physics with continuous symmetry: the case of singular interactions. *Commun. Math. Phys.*, **226** 2002, 433–454.
- [8] M. Kelbert, Y.Suhov. A quantum Mermin–Wagner theorem for a generalized Hubbard model on a 2D graph. Submitted to *Adv. Math. Phys.*; arXiv:1211.5446v2 [math-ph]
- [9] M. Kelbert, Y.Suhov , A. Yambartsev. A Mermin-Wagner theorem for Gibbs states on Lorentzian triangulations. To appear in *Journ. Statist. Phys.*; arXiv: 1210.7981 [math-ph]
- [10] M. Kelbert, Yu. Suhov, A. Yambartsev. A Mermin–Wagner theorem on Lorentzian triangulations with quantum spins. Submitted to *Brazilian Journ. Probab.*; arXiv:1211.5446 [math-ph]

- [11] Y. Kondratiev, Y. Kozitsky, T. Pasurek. Gibbs random fields with unbounded spins on unbounded degree graphs. *Journ. Appl. Probab.*, **47** 2010, 856–875
- [12] Y. Kozitsky, T. Pasurek. Euclidean Gibbs measures of interacting quantum anharmonic oscillators. *Journ. Stat. Phys.*, **127** 2007, 985–1047
- [13] N.D. Mermin, H. Wagner, H. Absence of ferromagnetism or antiferromagnetism in one- or two-dimensional isotropic Heisenberg models. *Phys. Rev. Lett.*, **17** 1966, 1133–1136
- [14] C.-E. Pfister. On the symmetry of the Gibbs states in two-dimensional lattice systems. *Commun. Math. Phys.*, **79** 1981, 181–188
- [15] M. Reed, B. Simon. *Methods of Modern Mathematical Physics. Vol. I: Functional Analysis*, Academic Press, 1972; *Vol. II: Fourier Analysis, Self-Adjointness*, Academic Press, 1975; *Vol. IV: Analysis of Operators*, Academic Press, 1977.
- [16] T. Richthammer. Two-dimensional Gibbsian point processes with continuous spin symmetries. *Stochastic Process. Appl.*, **115**, 2005, 827–848
- [17] T. Richthammer. Translation invariance of two dimensional Gibbsian point processes. *Commun. Math. Phys.*, **274**, 2007, 81–122
- [18] T. Richthammer. Translation invariance of two dimensional Gibbsian systems of particles with internal degrees of freedom. *Stoch. Process. Appl.*, **119**, 2009, 700–736
- [19] K.-I. Sato. *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press, 2011
- [20] B. Simon. *Functional Integration and Quantum Physics*. Academic Press, New York, 1979
- [21] B. Simon. *The Statistical Mechanics of Lattice Gases*. Princeton University Press, Princeton, NJ, 1983.
- [22] B. Simon, A. Sokal. Rigorous entropy-energy arguments. *Journ. Stat. Phys.*, **25** (1981), 679–694

- [23] Yu. M. Suhov. Existence and regularity of the limit Gibbs state for one-dimensional continuous systems of quantum statistical mechanics. *Soviet Math. (Dokl.)*, **11** (**195**), 1970, 1629–1632